INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 2, Pages 374–388

APPLICATION OF AN ENERGY-MINIMIZING ALGEBRAIC MULTIGRID METHOD FOR SUBSURFACE WATER SIMULATIONS

JING-RU C. CHENG, XUE-HAI HUANG, SHI SHU, JINCHAO XU, CHEN-SONG ZHANG, SHUO ZHANG, AND ZHIYANG ZHOU

Abstract. Efficient methods for solving linear algebraic equations are crucial to creating fast and accurate numerical simulations in many applications. In this paper, an algebraic multigrid (AMG) method, which combines the classical coarsening scheme by [19] with an energy-minimizing interpolation algorithm by [26], is employed and tested for subsurface water simulations. Based on numerical tests using real field data, our results suggest that the energy-minimizing algebraic multigrid method is efficient and, more importantly, very robust.

Key words. subsurface water simulation, multigrid, algebraic multigrid, energy-minimizing interpolation.

1. Introduction

Various mathematical models have been proposed to simulate the flow of water, sediment, chemicals, nutrients, and microbial organisms within watersheds, as well as to quantify the impact of human activities in the course. In these models, systems of partial differential equations (PDEs) are used to describe one or several physical processes of the subsurface water flow. As it is often impossible to obtain exact solutions, we turn to numerical simulations to help us understand the complicated physics underlying such processes.

Various discretization methods for PDEs (see [17, 7]) can be applied to reduce the continuous differential equations to finite dimensional sparse linear systems. There are many possible algorithms for solving sparse linear systems (see [20]) that arise from discretizations of PDEs, among which is the geometric multigrid (GMG) method. When applicable, GMG is generally considered one of the most efficient techniques (e.g., [1, 6, 25, 22] and [11]).

However, since GMG requires an explicit hierarchy of the underlying grids and its implementation is problem dependent, its applicability is limited in practice. The algebraic multigrid (AMG) method, on the other hand, requires minimal geometric information about the underlying problem and can sometimes be employed as a "black-box" solver for a range of problems. The classical AMG method ([3] and [18]) has been shown to be effective for a range of problems (cf. [19, 21] and [10]).

Since the early 1980s, considerable effort has been devoted to enlarging the applicability of AMG. Many different types of algebraic multigrid methods have been developed. In the interest of making implementation easier, [23, 13] and [16] all proposed aggregation-type AMG methods. The energy-minimizing interpolation approach ([24, 26]) constructs coarse-grid spaces by computation to enhance stability and approximation. Other examples of AMG methods include element interpolation based AMG (or AMGe) by [5, 12] and [8], element agglomeration based

Received by the editors February 28, 2012.
2000 *Mathematics Subject Classification.* 65F10, 65N22.
This author was partially supported by the Key Project of Scientific Research Fund of Hunan Provincial Science and Technology Department (Grant No. 2011FJ2011).

AMG by [14], and compatible relaxation based techniques (see [2, 9, 15] and [4]). Each has advantages for solving certain problems.

There are several well-established free and commercial software packages for algebraic multigrid methods. However, we have found that the problems inhering in subsurface water simulations are still extremely challenging in practice; see Section 2 for details. The main difficulties lie in heterogeneity and anisotropic coefficients with large jumps. For some of the packages we tested, it was necessary to tune their parameters for every single test problem in order to make the iterative solvers converge to a satisfactory tolerance.

In this paper, we employ an energy-minimizing AMG method to address a subsurface water simulation problem. This method utilizes the Ruge–Stüben strategy to choose coarse-grid variables and to establish the interpolation operator on each level, it uses an energy-minimizing (EM) approach. The main idea of EM interpolation was proposed by [24] and later explored by [26]. We use the algorithm given by [26]. Two essential components for the convergence of multigrid methods, namely stability and approximation, are taken into account when constructing coarse-grid spaces; and, numerical experiments show that this AMG method, referred to as EMAMG, is robust with respect to the parameters of the subsurface water simulation test problems under consideration here.

We compare EMAMG with the well-known classical AMG method—using the Ruge–Stüben strategy to choose coarse-grid variables and the direct interpolation for constructing grid transfer operators. The classical AMG method can be found in references including [18, 21], and [22]. Numerical experiments in the present paper show that, in our implementation, the EMAMG method outperforms the AMG based on the classical interpolation (or RSAMG in short) for discrete problems arising from subsurface flow modeling. Hence, EMAMG is a competitive alternative to RSAMG.

The remainder of the paper is organized as follows. A model for simulating subsurface water problems is described in Section 2. The two types of AMG methods (RSAMG and EMAMG) under consideration are introduced in Section 3. Numerical experiments for RSAMG and EMAMG are presented in Section 4. Finally, some concluding remarks and observations based on our experiments are given in Section 5.

2. A model problem for subsurface water systems

Many subsurface models have been established to describe the contaminant transportation through saturated–unsaturated porous media. WASH123D is one such model based on the first principles (see [27]). We take an important equation from WASH123D as the test problem and it is labeled as SD-5. SD-5 is a three-dimensional problem on a subdomain of the first draft of the South Florida Regional Engineering Model for Ecosystem Restoration (REMER). It covers most of the land area south of the Tamiami Trail in South Florida, which is bounded by the Tamiami canal and the C4 canal in the north, and by the shores of the Gulf of Mexico, Florida Bay, and Biscayne Bay (Fig. 1).

In WASH123D, the governing equation of subsurface density dependent flow through saturated–unsaturated porous media can be derived based on the mass conservation of water; i.e.,

$$
\left\{
\begin{array}{rl}
\frac{\rho}{\rho_0}\mathcal{F}\frac{\partial h}{\partial t} & = -\nabla\cdot(\frac{\rho}{\rho_0}\mathbf{V}) + \frac{\rho^*}{\rho_0}q \\
\mathbf{V} & = -\mathbf{K}\cdot(\frac{\rho_0}{\rho}\nabla h + \nabla z),
\end{array}
\right.
\tag{1}
$$