J. Partial Differential Equations
Vol. 5, *No.* 4 (1992), 28–36

A PRIORI ESTIMATES AND EXISTENCE OF POSITIVE SOLUTIONS TO QUASILINEAR ELLIPTIC EQUATIONS IN GENERAL FORM

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(Received June 29, 1989; revised Dec. 29, 1991)

Abstract In this paper we prove the existence of a positive solution to the following superlinear elliptic Dirichlet problem,

$$-\sum_{i,j=1}^{n} a_{ij}(x,u,Du)D_{ij}u = f(x,u,Du) \text{ in } \Omega, \quad u=0 \text{ on } \partial\Omega$$

where f satisfies certain growth conditions.
Key Words Elliptic equation; positive solution.
Classifications 35J20; 35J25.

1. Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with C^2 boundary, $n \geq 2$. In this paper we are concerned with the problem of finding a function u satisfying the elliptic boundary problem

$$\begin{cases} Lu = -\sum\limits_{i,j=1}^{n} a_{ij}(x,u,Du)D_{ij}u = f(x,u,Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where L is a uniformly elliptic operator, i.e., there exist positive constants λ, Λ so that

$$\lambda|\eta|^2 \leq a_{ij}(x,u,\xi)\eta_i\eta_j \leq \Lambda|\eta|^2 \tag{1.2}$$

for all $\eta \in \boldsymbol{R}^n$, $(x,u,\xi) \in \Gamma = \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n$.

For divergent elliptic equations, if it is the Euler equation of a differentiable functional, using the variational principle one can obtain two or more solutions, see [1, 9]. But in our situation the equation in (1.1) is not a Euler equation of some functional and the variational methods can't be applied.

We will use the *a priori* estimates combined with degree theory to prove the existence of a positive solution to (1.1). The main difficulty is to establish the *a priori* bound $\|u\|_{L^\infty}$ for solutions u of (1.1). Generally such estimate is not true, see [1],

[9]. But for positive solutions various estimates for $\|u\|_{L^\infty}$ have been obtained [2]–[4]. Recently Gu [8] proved such estimates for positive solutions of the problem

$$-\Delta u = f(x, u, Du) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.3}$$

where f satisfies the growth condition

$$\lim_{|u|+|\xi|\to\infty} f(x,u,\xi)/(|u|^\alpha + |\xi|^\beta) = 0 \quad \text{uniformly for } x \in \Omega$$

with $\alpha = (n+1)/(n-1)$, $\beta = (n+1)(n-2)/n(n-1)$ or $1 < \alpha = \beta < (n+1)/n$.

Here we use the "blow up" method to derive the *a priori* estimates for positive solutions of (1.1). Suppose there exist $a_{ij}^*(x)$, $f^*(x) \in C(\bar{\Omega})$ with $f^*(x) \geq \delta_0 > 0$ such that

(H_1) $$\lim_{u\to\infty} a_{ij}(x,u,\xi) = a_{ij}^*(x) \text{ uniformly for } (x,\xi) \in \Omega \times \mathbf{R}^n$$

(H_2) $$|f(x,u,\xi) - f^*(x)u^p| \leq \mu(|u| + |\xi| + |u|^p + |\xi|^{2p/(p+1)})$$

where $p \in (1, (n+2)/(n-2))$, $\mu(t) \geq 0$ is a nondecreasing function satisfying

$$\overline{\lim_{t\to 0}}\, \mu(t)/t = \mu_0 < \infty \tag{1.4}$$

$$\lim_{t\to\infty} \mu(t)/t = 0 \tag{1.5}$$

Assumption (H_2) implies by the Harnack inequality (see [7]) that, if $u \in W^{2,n}(\Omega)$ is a nonnegative solution of (1.1) and $u \not\equiv 0$, then $u > 0$ in Ω. In the following we always suppose $f(x,u,\xi) = 0$ for $u \leq 0$.

In Section 2 of this paper we prove the *a priori* estimates for solutions of (1.1). In Section 3 we deal with the existence. For later applications we quote two results in [4].

Proposition 1 *If $u \geq 0$ satisfies $\Delta u + u^p = 0$ in $\mathbf{R}^n$ with $1 < p < (n+2)/(n-2)$, then $u \equiv 0$.*

Proposition 2 *If $u \geq 0$ satisfies $\Delta u + u^p = 0$ in $\mathbf{R}^n \cap \{x_n > 0\}$ and $u = 0$ on $\{x_n = 0\}$ with $1 < p < (n+2)(n-2)$, then $u \equiv 0$.*

2. *A priori* estimates

In order to apply the topological degree theory to problem (1.1), we have to establish the L^∞ *a priori* estimates for solutions of the problem

$$\begin{cases} -t\Delta u + (1-t)Lu = tu_+^p + (1-t)f(x,u,Du) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1$_t$}$$

for all $t \in [0,1]$, where L is the elliptic operator introduced in (1.1).

Theorem 2.1 *If $u \in W^{2,n}(\Omega)$ is a nonnegative solution of (2.1), then*

$$\sup\{u(x); x \in \Omega\} < M \tag{2.2}$$

for some M independent of $t \in [0,1]$.