J. Partial Differential Equations
Vol. 4, *No.* 2 (1991), 1–18

CAUCHY PROBLEMS FOR DEGENERATE HYPERBOLIC MONGE-AMPERE EQUATIONS AND SOME APPLICATIONS

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(Received Sep. 11, 1989)

Abstract In this paper, the Cauchy problems for a class of nonlinear degenerate hyperbolic Monge-Ampere equations are studied and some results on local isometric embedding in R^3 of two dimensional Riemannian manifolds with non positive curvature, are contained.

Key Words degenerate hyperbolic; Monge-Ampere equation; local isometric embedding.
Classification 35L65.

1. Introduction

The present paper is devoted to the local isometric embedding in R^3 of two dimensional Riemannian manifolds with non positive curvature and to the global construction of surfaces with given curvature changing sign. So far there have been many papers for the local isometric embedding, for example [5] and [6] for the cases that the curvature K associated with a given metric is non negative and clearly changing sign, respectively, [8] for the case $K < 0$ and [4] for the case $K \leq 0$ and $d^2K \neq 0$. However the obtained results for the case $K \leq 0$ are not quite complete. Analytically speaking, the difficulty in attacking non positive case consists in the linearized equation of degenerate hyperbolic Monge-Ampere equation to be not effective if K is of severe degeneracy, and hence, generally speaking, the Cauchy problem for it is of illness. Let γ be a smooth curve in R^2 and let h be its defining function, i.e., $h = 0$ and $dh \neq 0$ on γ. Suppose that the curvature K associated with a given metric $g = g_{ij}du^i du^j$ satisfies.

$$K = h^q K_1 \text{ near a point } p \text{ on } \gamma \tag{1.1}$$

for some smooth positive function K_1 and a natural number q. If $q \geq 3$, the linearized equation of degenerate hyperbolic Monge-Ampere equation is most probably, non effectively hyperbolic. This is just the case we shall study in the present paper. Of course our discussion also coveres the cases $q = 2$ and $q = 1$.

Section 2 and Section 3 are concerned with one side Cauchy problem for the case that q is an arbitrary natural number. To avoid the difficulty mentioned above we carefully choose initial data so that the corresponding linearized problems are always well-posed in utilizing Nash-Moser-Hormander procedure. A priori estimates for such linearized problems are obtained with the aid of the straightforward way as done in [1].

As applications of the result obtained in Section 3, two main theorems which are concerning the local isometric embedding and other geometric problems are proved in Section 4. The first one is as follows.

Theorem A *Suppose the curvature K associated with a given metric g satisfies* (1.1) *with $q=2\bar{q}$ for some natural number $\bar{q}$. Then the local isometric embedding has a C^∞ solution near the point p.*

Another theorem is closely related to global construction of a graph with a given curvature changing sign clearly, i.e., $q = 1$ in (1.1). Let γ be the boundary of a bounded smooth convex domain Ω in R^2 and let K be a smooth function defined in a neighbourhood of $\Omega, N(\Omega)$.

$$\begin{aligned} &K(x) > 0 \quad \text{in } \Omega,\ K(x) < 0 \quad \text{in } N(\Omega) \setminus \bar{\Omega} \\ &\text{and } dK \neq 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

Let us consider

$$\det(u_{ij}) = K(x)F(x, u, \nabla u), \quad x \in N(\Omega) \tag{1.3}$$

with

$$u = 0, \text{ on } \partial\Omega \tag{1.4}$$

Evidently, for such a function $K(x)$ satisfying (1.2), (1.3) is of mixed type if F is positive. We also assume that $F(x, u, p)$ is subject to the following conditions:

$$F \text{ is convex in } p \text{ and } F_u \geq 0 \text{ in } \bar{\Omega} \times R^1 \times R^2 \tag{1.5}$$

$$F(x, N, p) \leq h^{-1}(p) \text{ for all } x \text{ in } \bar{\Omega} \text{ and } p \text{ in } R^2 \tag{1.6}$$

$$F(x, 0, p) \leq C(1 + p^2)^2 \tag{1.7}$$

for some nonnegative constants C, N and a positive function h in $L^1_{\text{loc}}(R^2)$ satisfying a structure conditions

$$\int_\Omega K dx < \int_{R^2} h dp \tag{1.8}$$