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SOME PROPERTIES OF FREE BOUNDARY FOR SOLUTION OF POROUS MEDIUM EQUATION WITH GRAVITY TERM IN ONE-DIMENSION

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Abstract Consider the degenerate parabolic equation (porous medium equation with gravity term):

$$u_t = (u^m)_{xx} + (u^n)_x, \ -\infty < x < \infty, \ t > 0, m > 1$$
$$u(x,0) = u_0(x), \ -\infty < x < \infty$$

The main results consist of the estimation of t_i^* called waiting time, the behavior of pressure $V = \dfrac{m}{m-1}u^{m-1}$ near a vertical or a nonvertical part of $\varsigma_i(t)$ and a condition of that $\varsigma_i(t)$ is continuously differentiable.

Key Words porous medium equation; gravity term; free boundary; C^1 regularity; waiting time.
Classification 35K65.

1. Introduction

Consider the Cauchy problem:

$$u_t = (u^m)_{xx}, \ -\infty < x < \infty, \ t > 0 \tag{1.1}$$

$$u(x,0) = u_0(x), \ -\infty < x < \infty \tag{1.2}$$

It is well known to us that if initial datum $u_0(x)$ is a nonnegative, bounded measurable function in $(-\infty, \infty)$ and has compact support, i.e.

$$u_0(x) > 0, \text{ if and only if } x \in (a_1, a_2)$$

for $-\infty < a_1 < a_2 < \infty$, then there exist two continuous curves $x = \varsigma_i(t)$ $(i = 1, 2)$ satisfying

$$P[u] \equiv \{(x,t) : u(x,t) > 0, -\infty < x < \infty, t > 0\}$$
$$= \{(x,t); \varsigma_1(t) < x < \varsigma_2(t); t > 0\}$$
$$\varsigma_i(0) = a_i$$

where $u(x,t)$ is the solution of (1.1), (1.2). $\varsigma_i(t)(i = 1,2)$ is called the free boundary of nonnegative weak solution $u(x,t)$ for the porous medium equation. Moreover $(-1)^i\varsigma_i(t)$ is a nondecreasing function and has uniformly Lipschitz's continuity in $[\tau, \infty)$ for any $\tau > 0$.

D. G. Aronson, L. A. Caffarelli & S. Kamin[1] and Knerr[2] have made systematic and profound studies on the properties of $\varsigma_i(t)$. Chen Yazhe[3] has recently studied the waiting time for general degenerate parabolic equation in the case of several space variables.

In this paper, we study the Cauchy problem for the porous medium equation with gravity term in one-dimension:

$$u_t = (u^m)_{xx} + (u^n)_x, \ -\infty < x < \infty, \ t > 0, m > 1 \tag{1.3}$$

$$u(x,0) = u_0(x), -\infty < x < \infty \tag{1.4}$$

In the papers [1] and [2], the main techniques are: estimating lower bound for V_{xx} in $P[u]$ (where the pressure $V = \dfrac{m}{m-1}u^{m-1}$), carrying similarly transformation for the solution of the pressure equation which is satisfied by V (see (1.6)) and constructing some explicit special solutions of the pressure equation for comparison. Because of adding the gravity term, there is somewhat substantial difficulties in generalizing their method to the equation (1.3). On the basis of [4], we have now further overcome some difficulties and generalized the main results for the equation (1.1) to the equation (1.3).

We set

$$S_T = (-\infty, \infty) \times (0, T], \ S = (-\infty, \infty) \times (0, \infty)$$

Definition 1 *A function $u(x,t)$ defined on $\bar{S}_T$ is said to be a weak solution of problem(1.3), (1.4) in $\bar{S}_T$, if*

(i) *u is nonnegative, bounded and continuous on $\bar{S}_T$.*

(ii) *The generalized derivative $(u^m)_x \in L^2(S_T)$.*

(iii) *u satisfies the identity*

$$\iint_{S_T} \{\phi_x[(u^m)_x + u^n] - \phi_t u\}dxdt = \int_{-\infty}^{\infty} \phi(x,0)u_0(x)dx \tag{1.5}$$