J. Partial Differential Equations
Vol. 4, *No.* 2 (1991), 36–44

SOLUTIONS OF ELLIPTIC EQUATIONS $\Delta u + K(x)e^{2u} = f(x)$

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(Received April 29, 1989)

Abstract In this paper we consider the elliptic equation $\Delta u + K(x)e^{2u} = f(x)$, which arises from prescribed curvature problem in Riemannian geometry. It is proved that if $K(x)$ is negative and continuous in R^2, then for any $f \in L^2_{\rm loc}(R^2)$ such that $f(x) \le K(x)$, the equation possesses a positive solution. A uniqueness theorem is also given.
Key Words elliptic equations; prescribed curvature problem; monotone operators; Kato's inequality.
Classifications 35B; 47H.

Let (M, g_0) be a 2-dimensional Riemannian manifold, with Gaussian curvature $K_0(x)$, and let $K(x)$ be a given function on M. The prescribed curvature problem is as follows: can we find a new metric g on M, such that g is conformal to g_0 and with Gaussian curvature $K(x)$? This problem is equivalent to solvability for the following elliptic equation:

$$\Delta u - K_0(x) + K(x)e^{2u} = 0, \quad x \in M \tag{1}$$

where Δ is Laplacian on (M, g_0).

In the case of compact manifolds, this problem has been studied in detail, see [1–3]. In [2], this problem was posed on complete, non-compact Riemannian manifolds.

In the case $M = R^2$, equation (1) reduces to

$$\Delta u + K(x)e^{2u} = 0, \; x \in R^2 \tag{2}$$

Many existence and non-existence results for equation (2) were proved, see [4–8].

Sattinger [4] proved that if $K(x) \le 0$ for all x and if $K(x) \le -\dfrac{c}{|x|^2}$ for $|x|$ large, then (2) possesses no solution.

Ni [5] proved the first existence theorem: if $K(x) \le 0$ for all x and if $K(x) \ge -\dfrac{c}{|x|^l}$ for $|x|$ large and some $l > 2$, then equation (2) possesses infinitely many solutions.

The generalized equation was considered in [6–8]:

$$\Delta u + K(x)e^{2u} = f(x), \quad x \in R^2 \tag{3}$$

Kenig-Ni [6] proved that if $K(x) \le 0$ for all x, and

$$|f(x)| \le \frac{c}{|x|^{2+\varepsilon}}, \quad |K(x)| \le \frac{c}{|x|^{2+\alpha}} \qquad \text{for } |x| \text{ large}$$

where $\varepsilon > 0$, $\alpha > \frac{1}{\pi}\int_{R^2} f(x)dx$, then equation (3) possesses infinitely many solutions. For non-existence, they proved that if $f(x) \ge 0$ and $K(x) \le 0$ for all x, and

$$f(x) \ge \frac{c_0}{|x|^{2+\alpha}} \quad \text{for } |x| \ge R_0$$

$$|K(x)| \ge \frac{c}{|x|^{2+\alpha}} \quad \text{for } |x| \text{ large, and some } \alpha < \frac{2c_0}{\varepsilon R_0^{\varepsilon}}$$

then (3) possesses no solution.

McOwen [7] proved that if $K(x) \le 0$ for all x and $K(x) < 0$ on support of $f(x)$, and

$$f(x), K(x) = O(|x|^{-l}) \quad \text{for } |x| \to \infty$$

where $l > 2$, then (3) possesses a bounded solution which is asymptotic to some constant.

Hong [8] proved several related results.

In the papers mentioned above, the decay condition $K(x) = O(|x|^{-l})$ for $|x| \to \infty$ is essential. What we are interested in is the following question: If $K(x)$ does not satisfy the decay condition, for which kind of functions f's can equation (3) still have solutions?

In the following we only consider generalized solutions for equation (3), that means, $u \in L^1_{\text{loc}}(R^2)$ such that $K(x)e^{2u} \in L^1_{\text{loc}}(R^2), \Delta u \in L^1_{\text{loc}}(R^2)$, and (3) holds almost everywhere, and for any $\varphi \in C_0^\infty(R^2)$,

$$\int_{R^2} u \cdot \Delta\varphi dx + \int_{R^2} K(x)e^{2u}\varphi dx = \int_{R^2} f\varphi dx$$

If u is a generalized solution of (3), then we write the above equality as

$$\Delta u + K(x)e^{2u} = f(x) \quad \text{in } D'(R^2) \tag{4}$$

The main result of this paper is the following:

Theorem 1 *Let $K(x)$ be negative and continuous in R^2, then for any function $f \in L^2_{\text{loc}}(R^2)$, $f(x) \le K(x)$ for all $x \in R^2$, equation (4) possesses a non-negative solution $u \in W^{1,2}_{\text{loc}}(R^2)$. If in addition $f(x) \ne K(x)$, then u is positive everywhere.*

The idea of proof comes from [9], where Brezis used Kato's inequality [10] and Baras-Pierre's technique [11] to prove the existence and uniqueness of generalized solutions for the equation

$$-\Delta u + |u|^{p-1}u = f(x) \quad \text{in } R^N$$

where $p > 1$. Also Gallouet-Morel [12] used Brezis' idea to discuss the case $0 < p < 1$.

The following formula of Kato's inequality appeared in [9, Lemma A1].