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STRONG SOLUTION OF THE OBSTACLE PROBLEM FOR FULLY NONLINEAR ELLIPTIC EQUATIONS

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Abstract In this paper we obtain the existence of $W^{2,\infty}$ solutions of the obstacle problems for fully nonlinear elliptic equations under more general structure conditions than those in [1] by using the mollifier approach, which is also extended in our discussion.
Key Words Strong solution; fully nonlinear elliptic equations; the mollifier approach.
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1. Introduction

In [1], by using the method of viscosity solution given by P. L. Lions in [2], Hu Bei deals with the obstacle problem for fully nonlinear elliptic equations

$$\begin{cases} -F(x,u,Du,D^2u) \leq 0 & \text{in } \Omega \\ u(x) \leq g(x) & \text{in } \Omega \\ (u-g)F(x,u,Du,D^2u) = 0 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

He proves that the above problem has the unique viscosity solution in $W^{1,\infty}(\Omega) \cap C^{2,\alpha}(K_u)$ under the natural structure conditions and the concave condition as in [3], where

$$K_u = \{x \in \Omega |\ u(x) < g(x)\} \tag{1.2}$$

However, while he discussed the strong solutions, in view of the restriction of his method, he studies only the following special form

$$F(x,u,p,r) = F(r) + f(x,u,p) \tag{1.3}$$

where F and f are concave in r and p respectively.

The purpose of this note is to get the strong solution of (1.1) under the general structure conditions. We give the new method of $W^{2,\infty}$ estimate for solutions here.

Let Ω be a bounded open domain in R^n and let $\Gamma = \Omega \times R \times R^n \times S^n$, where S^n is the $n(n+1)/2$-dimensional space made of symmetric matrices of n-order. Assume that the obstacle $g(x)$ is in $C^2(\bar{\Omega})$ and satisfies the consistent condition

(G1) $g(x) \geq 0$ on $\partial\Omega$

and that $F(x,z,p,r)$ in Γ satisfies the following structure conditions:

(F1) There exists positive constant $\lambda > 0$ such that

$$\lambda|\xi|^2 \leq \frac{\partial F}{\partial r_{ij}}\xi_i\xi_j \leq \lambda\mu_1(|z|)|\xi|^2, \quad \forall\xi \in R^n$$

(F2) $|F(x,z,p,0)| \leq \lambda\mu_2(|z|)(1+|p|^2)$

(F3) $(1+|p|)^{-1}|F_x| + |F_z| + (1+|p|)|F_p| \leq \lambda\mu_3(|z|)(1+|p|^2+|r|)$

$|F_{xx}| + |F_{zx}| + |F_{pz}| + |F_{zz}| + |F_{zp}| + |F_{pp}| + (1+|r|)(|F_{rx}| + |F_{rz}| + |F_{rp}|)$

$\leq \lambda\mu_4(|z|,|p|)(1+|r|)$

(F4) $F(x,z,p,r)$ is concave in r.

(F5) $|F(x,z,p,r) - F(x_0,z_0,p_0,r)|$

$\leq \lambda\mu_5(|z|,|z_0|,|p|,|p_0|)\{|r|(|x-x_0|+|z-z_0|)+|r|^{1-\beta}+1\}$

(F6) $F(x,z,p,r)$ is strictly decreasing in z.

(F7) $F_z(x,z,0,0) \leq -\lambda\mu_0$

(F7)′ $F(x,g(x)+z,\ Dg(x)+p,\ D^2g(x))\mathrm{sgn}z \leq \lambda\mu_0(1+|p|)$

where $\beta \in (0,1)$ and $\mu_i (i=1,2,3,4,5)$ are non-decreasing in their variables and $\mu_0 > 0$ is a constant.

The main results in this paper are the following:

Theorem 1.1 *Suppose that $g(\cdot) \in C^2(\bar\Omega)$ satisfies (G1) and that F satisfies (F1)–(F6), (F7) or (F7)′. Then the obstacle problem (1.1) has the unique solution in $W^{1,\infty} \cap W^{2,\infty}_{\mathrm{loc}}(\Omega)$.*

Theorem 1.2 *Suppose that $\partial\Omega \in C^2$ and*

(G2) $g(x) > 0$ on $\partial\Omega$ or $g(x) \equiv 0$ on $\partial\Omega$

Under the conditions of Theorem 1.1, the obstacle problem (1.1) has the unique solution in $W^{2,\infty}(\Omega)$.

For simplicity, we always suppose $\lambda = 1$.

2. Preliminary Lemmas

First we often need the following well-known lemma:

Lemma 2.1 (cf [4], Lemma 3.1 on p.161) *Let $\varphi: [T_0,T_1] \to R^+$ be a bounded function, where $T_1 > T_0 \geq 0$. If for any $s,t: T_0 \leq t < s \leq T_1$, φ satisfies*

$$\varphi(t) \leq \theta\varphi(s) + A + \frac{B}{(s-t)^\alpha} \tag{2.1}$$

where $\theta \in (0,1)$ and A, B, α are non-negative constants. Then

$$\varphi(t) \leq C\Big[A + \frac{B}{(s-t)^\alpha}\Big] \quad for\ all\ T_0 \leq t < s \leq T_1 \tag{2.2}$$

where C depends only on θ and α.

In what follows we shall show the fact that the Hölder norm of a function can be described by its mollification, which is first given by N. S. Trudinger (cf [7]). Here we give some further properties to meet our need.