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INITIAL-BOUNDARY VALUE PROBLEM FOR A DEGENERATE QUASILINEAR PARABOLIC EQUATION OF ORDER $2m$ ①

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Abstract In this paper we consider the initial-boundary value problem for the higher-order degenerate quasilinear parabolic equation

$$\frac{\partial u(x,t)}{\partial t}+\sum_{|\alpha|\leq M}(-1)^{|\alpha|}D^\alpha A_\alpha(x,t,\delta u,D^m u)=0$$

Under some structural conditions for $A_\alpha(x,t,\delta u,D^m u)$, existence and uniqueness theorem are proved by applying variational operator theory and Galĕrkin method.

Key Words Higher-order degenerate equation; semibounded-variational operator; Galĕrkin method.

Classifications 35K35; 35K65.

1. Introduction

Let Ω be a bounded domain in R^n, $Q=\Omega\times(0,T]$. Consider the following initial—boundary value problem for the parabolic equation of order $2m$:

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}+\sum\limits_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha A_\alpha(x,t,\delta u,D^m u)=0, & (x,t)\in Q\\ \delta u=(u,Du,\cdots,D^{m-1}u) & \\ \delta u=0, \quad (x,t)\in\partial\Omega\times(0,T], & \\ u(x,0)=u_0(x), \quad x\in\Omega, u_0(x)\in L^2(\Omega) & \end{cases} \tag{1}$$

When $A_t(u)=\sum\limits_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha A_\alpha(x,t,\delta u,D^m u)$ for each t in $[0,T]$ is a regular elliptic operator in the Sobolev space $W^{m,p}(\Omega)$, the problem (1) had been considered by [1]—[3]. In this paper we discuss initial-boundary value problem (1) for a weak degenerate equation. This is the generalization of a result obtained by the writers for the equation of second order (see [4]).

First we introduce the fundamental space V which denotes the completion of $\mathring{C}^m(\Omega)=\{\varphi(x,\cdot)\in C^m(\bar\Omega)$; which vanish on a neighborhood of $\partial\Omega\}$ with respect to norm

$$\|\varphi(x,\cdot)\|_V=\Big\{\sum_{|\alpha|=m}\|\lambda(x)D^\alpha\varphi(x,\cdot)\|^p_{L^p(\Omega)}\Big\}^{1/p}$$

where $\lambda(x)\in L^p(\Omega)$, $\lambda^{-1}(x)\in L^s(\Omega)$, $s>n>1$, $\frac{1}{s}+\frac{1}{p}=\frac{1}{q}<1$, $2\leq p\leq\frac{ns}{s-n}$.

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Now we introduce $L^p(0,T;V)$ which is a space of functions defined on $[0,T]$ with values in V such that $\left\{\int_0^T \| u(x,t) \|_V^p dt\right\}^{1/p} < +\infty$.

Lemma 1.1 *V is a separable and reflexive Banach space (cf. [5]).*

Lemma 1.2 *V is continuously embedded in $\dot{W}^{m,q}$, and $\dot{W}^{m,q}$ is compactly embedded in $\dot{W}^{m-1,\sigma}$, where σ satisfies $p<\sigma<\frac{nq}{n-q}$, and $q=\frac{ps}{p+s}$.*

Proof It is sufficient to point out that the following inequality can be established for function $u(x,\cdot)\in \dot{C}^m(\Omega)$

$$\| D^m u \|_{L^q} = \left\{\int_\Omega |\lambda^{-1}(x)\lambda(x)D^m u|^q dx\right\}^{1/q} \leqslant \| \lambda^{-1}(x) \|_{L^s} \| \lambda(x) D^m u \|_{L^p}$$

By the interpolation inequality, $\left\{\sum_{|\alpha|=m} \| D^\alpha u \|_{L^q}^q\right\}^{1/q}$ would be an equivalent norm on $\dot{W}^{m,q}$, then the first conclusion of the lemma is obtained. By the compact imbedding theorem it is easy to show that $\dot{W}^{m,q}$ is compactly embedded in $\dot{W}^{m-1,\sigma}$ if σ satisfied $p<\sigma<\frac{nq}{n-q}$.

Lemma 1.3 *If $u(x,t)\in L^p(0,T;V)$ and $u'(x,t)=\frac{\partial u(x,t)}{\partial t}\in L^{p'}(0,T;V^*)$, then $u(x,t)\in C^0(0,T;L^2(\Omega))$* (cf. [3]).

Definition *A function $u(x,t)\in L^p(0,T;V)\cap C^0(0,T;L^2(\Omega))$ is called a generalized solution of problem (1) if $u'(x,t)\in L^{p'}(0,T;V^*)$, $u(x,0)=u_0(x)$, and $u(x,t)$ satisfies*

$$\int_0^T \langle u',v\rangle dt + \int_0^T A_t(u,v)dt = 0, \quad \forall\, v(x,t) \in L^p(0,T;V)$$

where

$$\langle u',v\rangle = \int_\Omega u'(x,\cdot)v(x,\cdot)dx$$

$$A_t(u,v) = \sum_{|\alpha|\leqslant m}\int_\Omega A_\alpha(x,t,\delta u, D^m u)D^\alpha v(x,\cdot)dx \tag{2}$$

Furthermore we assume that $A_\alpha(x,t,\zeta,\xi)$ are Caratheodory functions and satisfy the following structural conditions:

Structural condition I

$$\sum_{|\alpha|=m} |A_\alpha(x,t,\zeta,\xi)\eta_\alpha| \leqslant a_0 \sum_{\substack{|\alpha|=|\beta|=m\\|l|\leqslant m-1}} |\lambda(x)\eta_\alpha|(|\lambda(x)\xi_\beta|^{p-1} + |\zeta_l|^{\sigma/p'} + |a_1(x)|)$$

$$\sum_{|\alpha|\leqslant m-1} |A_\alpha(x,t,\zeta,\xi)| \leqslant b_0 \sum_{\substack{|\beta|=m\\|l|\leqslant m-1}} (|\lambda(x)\xi_\beta|^{p/\sigma'} + |\zeta_l|^{\sigma/\sigma'} + |b_1(x)|)$$

where $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{\sigma}+\frac{1}{\sigma'}=1$, $a_1(x)\in L^{p'}(\Omega)$, $b_1(x)\in L^{\sigma'}(\Omega)$. The condition $\lambda^{-1}(x)\in L^s(\Omega)$ implies that the equation in (1) is weakly degenerate.

Remark There is a relation between the growth factors p and σ:

$$\frac{\sigma}{\sigma'} > \frac{\sigma}{p'} > \frac{p}{\sigma'} > p-1$$