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THE UNIQUENESS OF STEADY-STATE SOLUTION FOR TWO-PHASE CONTINUOUS CASTING PROBLEM

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Abstract Concerning steady-state continuous casting problem, we know that if the number of phases is one, both existence and uniqueness had been solved ([1],[2],[3]), if the number of phases is two, the existence had been proved ([4]), but the uniqueness of weak solution is an open problem all the time. This paper is devoted to solving this problem.

Key Words Partial differential equation; free boundary problem; uniqueness.
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1. The Steady-state Continuous Casting Problem

The portion of ingot considered is supposed to include the solid-liquid interface (Figure) and occupies a cylindrical open domain $\Omega=\Gamma\times(0,H)$ of R^n ($\Gamma=(0,a)$ for $n=2$ and Γ is an open bounded domain of R^2 with Lipschitz boundary for $n=3$.)

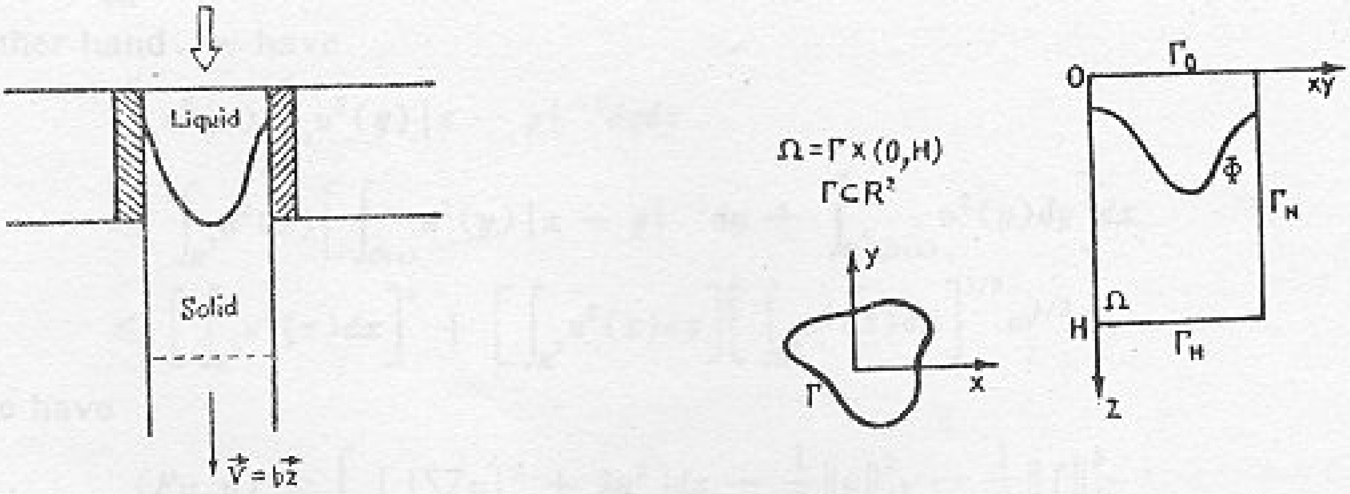

Figure (a) Ingot solidification in continuous casting (b) Ingot geometry in R^n

We set $\Gamma_i=\Gamma\times\{i\}, i=0,H$, $\Gamma_D=\Gamma_0\cup\Gamma_H$ and $\Gamma_N=\partial\Gamma\times(0,H)$, we denote the gradient by $\nabla=(\partial_x,\partial_y,\partial_z)$, so $\Delta=\nabla\circ\nabla$. We shall assume free boundary $\Phi=\{(x,y,z)\in\Omega: z=\Phi(x,y)\}$ fixed with respect to the mould and the casting velocity given by $\vec{v}=b\vec{z}$ with constant $b>0$. The metal temperature $T=T(x,y,z)$ verifies stationary heat equation

$$bC(T)\partial_z T=\nabla(k(T)\nabla T) \quad \text{in} \quad \Omega\backslash\Phi \tag{1}$$

where $C\geq 0$ is the specific heat and $k>0$ the thermal conductivity. The left member in (1) takes into account the heat transfer due to the convection. If T_0 denotes the melting temperature at the interface, after the usual renormalization procedure

$$\theta = \int_T^{T_0} k(\tau)d\tau = K(T) \qquad (2)$$

at the solid region $\{\theta > 0\}$ and at the liquid region $\{\theta < 0\}$, equation (1) becomes

$$\partial_z f(\theta) = \Delta\theta \quad \text{in} \quad \Omega\backslash\Phi = \{\theta > 0\} \cup \{\theta < 0\} \qquad (3)$$

where $f = C_b \circ K^{-1}$ and $C_b(T) = b\int_T^{T_0} C(\tau)d\tau$. At the interface we have $\theta = 0$ and the stefan condition is given, in terms of the renormalized temperature θ, by

$$-[\nabla\theta]_-^+ \circ \vec{\gamma} = -\lambda\vec{v} \circ \vec{\gamma} = \lambda b \quad \text{on} \quad \Phi = \{\theta = 0\}$$

where $\lambda > 0$ is the latent heat, $\vec{\gamma} = (\partial_x\Phi, \partial_y\Phi, -1)$ is a normal vector to Φ and $[\ ]_-^+$ denotes the jump across Φ.

2. Definition of Weak Solution Existence of Solution to the Two-phase Problem

Problem (P) Find a couple $(\theta, \eta) \in H^1(\Omega) \times L^\infty(\Omega)$, such that

$$\theta = h \quad \text{on} \quad \Gamma_D \qquad (4)$$

$$0 \le \chi\{\theta > 0\} \le \eta \le 1 - \chi\{\theta < 0\} \le 1 \quad \text{a.e. in} \quad \Omega \qquad (5)$$

$$\int_\Omega \{\nabla\theta\nabla\zeta - [f(\theta) + \lambda b\eta]\partial_z\zeta\} + \int_{\Gamma_N} g(x,y,z,\theta)\zeta = 0$$

$$\forall \zeta \in H^1(\Omega); \zeta = 0 \quad \text{on} \quad \Gamma_D \qquad (6)$$

For our existence result, we shall assume that $f = f(\theta): R \to R$ is a continuous function; $g = g(x,y,z,\theta): \Gamma_N \times R \to R$ is a Caratheodory function, i.e., it is measurable in $(x,y,z) \in \Gamma_N$ for all $\theta \in R$ and continuous in θ for a.e. (x,y,z). Furthermore, letting μ and M be given constants, for a.e. $(x,y,z) \in \Gamma_N$, we assume

$$g(x,y,z,\theta)\theta \ge 0 \quad \text{if} \quad \theta \le \mu < 0 \quad \text{or} \quad \theta \ge M > 0$$

$$\forall L > 0, \exists\, \bar{g}_L \in L^q(\Gamma_N), \quad q > n-1, \text{such that}$$

$$|g(x,y,z,\theta)| \le \bar{g}_L(x,y,z), \quad \text{for} \quad |\theta| < L;$$

$$h \in C^{0,1}(\bar{\Gamma}_D), \mu \le h|_{\Gamma_0} < 0 \quad \text{and} \quad 0 < h|_{\Gamma_H} \le M$$

Under preceding conditions for f, g and h, Rodrigues had proved that there exists a solution $(\theta, \zeta) \in H^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega}) \times L^\infty(\Omega)$ for some fixed $0 < \alpha < 1$ (see Theorem 1 in [4]).

3. Proof of Uniqueness of Weak Solution with Linear Cooling

We shall assume

$$f \in C^0(R) \cap C^1(R\backslash\{0\}) \quad , \quad \beta_1 \ge f' \ge \beta_2 > 0 \quad \text{in} \quad (R\backslash\{0\}) \qquad (7)$$

$$g(x,y,z,\theta) = \gamma(\theta - \rho) \quad \text{on} \quad \Gamma_H \qquad (8)$$

$$h \in C^{0,1}(\bar{\Gamma}_D) \quad \mu \le h < 0 \text{ on } \Gamma_0 \text{ and } 0 < h \le M \text{ on } \Gamma_H \qquad (9)$$

$$\rho \in L^\infty(\Gamma_N), \quad \mu \le \rho \le M \qquad (10)$$