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ESTIMATES FOR SURFACES WHICH ARE STATIONARY FOR AN ELLIPTIC PARAMETRIC INTEGRAL

Fang Hua Lin

(Courant Institute of Mathematical Sciences, New York Univ.)

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0. Introduction

In this paper we study an immersed surface M in R^3 which is stationary or stable for an elliptic parametric integral ψ. Although the proofs can be generalized to the case where M lies in a three dimensional manifold N, we find it more convenient to work just in R^3.

Under the hypothesis that the surface M is an embedded disks, Schoen and Simon[15] proved a uniform interior Hölder estimate of the unit normal (hence a curvature estimate when $\psi \in C^{3,\alpha}$). Under the same hypothesis, the author was able to prove the same estimate up to boundary provided $\partial M = \Gamma$ is a $C^{2,\alpha}$, Jordan curve, see[10]. By combining serval arguments of Choi-Schoen[4,10], J Pitts and I had observed a similar estimate for embedded surface with finite topology.

Embedded surface are very special compared to immersed ones. One can show easily, by various examples, that such estimates are no longer valid for an immersed surface M. Even in the case of minimal surfaces, some additional hypothesis are necessary in order to obtain certain curvature estimates. In [13], R. Schoen proved the curvature estimate for stable minimal surfaces in a three dimensional manifold. Schoen's proof uses intrinsic geometry and the stability inequality. There is a precise stability range for which Schoen's proof will work. The latter fact was verified by R. Gulliver and the author in 1984, see[7].

By taking the advantage of the second variation formula for ψ-integrals, which is due to W. Allard[1], and by combining the result of [7], one sees immediately that there is a positive constant ε (which is not too small) such that any connected, oriented, complete immersed surface in R^3 which is stable for an elliptic parametric integral ψ with constant is a plane provided that $\|\psi-1\|_{C^{2,\alpha}(S^2)} \leqslant \varepsilon$.

Here we use the Gauss-map to show some curvature estimates for general ψ-stationary or ψ-stable surfaces. The outline of the paper is as follows. In Section 1 we prove some preliminary estimates for a surface M having a bound on the square integral of the second fundamental form. In Section 2, a uniform interior curvature estimates is shown to be valid for such M's which satisfy also a weak-type stability hypothesis. It includes particularly the curvature estimates for ψ-stable surfaces. Under the condition that a minimal surface M satisfies a strong-type stability hypothesis, we prove the same estimate up to the boundary. Although the strong-type stability hypothesis is technical, it is necessary for the validity of the uniform curvature estimate. We give an example of a smooth family of analytic Jordan curves, which converges to a piecewise analytic Jordan curve, and which bounds a family of stable minimal disks which converges to a minimal disk spanning the limiting curve, such that the limit minimal disk is singular at a point which lies on an analytic portion of the boundary (i. e., the curvature is not bounded near this point). Such an example shows the falseness of the curvature estimate for minimal stable disks bounded by analytic Jordan curves. The main obstacle to our proof is the possible occurrence of branch points at the boundary. It remains unknown whether or not a least area disk bounded by a smooth Jordan curve possesses a boundary branch point. We study, in Section 4, the behavior of a least area disk near a boundary branch point (by assuming that there is one). A precise asymptotic expansion formula is obtained. As a consequence we show the continuity of Gauss-curvature for such a branched least area disk. An example of a branched minimal disk, due to R. Gulliver, shows that the behavior that we have described for least area disks near a possible boundary branch point does occurs for at least some branched minimal surfaces.

After the present work was essentially completed, the author received two interesting preprints by B. White [16, 17]. He proved various curvature estimates for general ψ-stationary surfaces in a three dimensional manifold which contains the estimates in [10, 11] and Section 1 of the present work. His proof is indirect. That is, he studies the behavior of complete surfaces in R^k having finite square integral of the second fundamental form, see [16]. Then the local estimate follows by the blow-up method. Our proof is direct and depends on the generalized Lebesgue's Lemma, see [10] or Section 1 below.

The materials in Section 4 have been reported on the Differential Geometry Seminar at the University of Minnesota in 1983. The author wishes to thank his advisor Professor R. Hardt for his encouragement and to Professor R. Gulliver and J. C. C. Nitsche for several interesting discussions.

This work was complete in the summer of 1986 while the author was visiting the Australian National University. It has been circulated in the Research Report of ANU (CMA-R28-86). Due many reasons it has not been sent for publication till now.

1. Preliminaries

Throughout this paper we shall adopt the following notations:

$M =$ a C^2-immersed surface in R^3;

$\psi =$ a 2-dimensional elliptic parametric integral of class $C^{3,\alpha}$ (see [12, chp. 9] for the discussion);

$B_\rho(\xi) = \{x \in R^3, |x - \xi| < \rho\}$;

$M_\rho(\xi) = M \cap B_\rho(\xi)$, for $\xi \in M$;