J. Partial Differential Equations
Vol. 3, *No.* 4(1990), 1—16

ON MODIFIED TRICOMI PROBLEM OF SEMI-LINEAR EQUATION OF MIXED TYPE OF SECOND KIND①

Xiao Zhuang
(Huizhou Teacher's College, Huangshan)
(Received May 10, 1988)

Abstract In this paper we considered the semi-linear equation of mixed type of second kind:

$$k(x,y)u_{yy}+u_{xx}+\alpha(x,y)u_y+\beta(x,y)u_x+\gamma(x,y)u-|u|^\rho u=f(x,y)$$

For the above equation, we solved the modified Tricomi problem and have proved the existence and uniqueness of strong solution in H_1.

Key Words equation of mixed type; modified Tricomi problem; strong solution
Classification 35M05.

0. Introduction

We consider the semi-linear equation of mixed type

$$Lu\equiv k(x,y)u_{yy}+u_{xx}+\alpha(x,y)u_y+\beta(x,y)u_x+\gamma(x,y)u-|u|^\rho u =f(x,y) \tag{1}$$

in a bounded simply connected domain $\mathscr{D}(=\mathscr{D}^+\cup\mathscr{D}^-)$, where the function $k(x,y)$ satisfies the conditions: $yk>0$ for $y\neq 0$, $k(x,0)=0$, $k\in C^2(\overline{\mathscr{D}})$; $\alpha,\beta,\gamma\in C^1(\overline{\mathscr{D}})$; $f\in L_2(\mathscr{D})$; ρ is a positive constant. The outer boundary of $\mathscr{D}^+-\Gamma_0$ is a picecwise smooth curve lying in the upper half-plane $y>0$ and intersecting the degenerate line $y=0$ at two points A and B. The outer boundary of $\mathscr{D}^--\Gamma_\pm: dy\pm\sqrt{-k}dx=0$ are a pair of characteristics of the equation (1), which emanate from A and B respectively. And so, Γ_+ and Γ_- also meet at the point R.

For the equation (1) we consider the following boundary value problem

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① This paper represents the results obtained during the author's studies at the Institute of Applied Physics and Computational Mathematics, Beijing, in 1988.

$$u = 0, \quad \text{on } \Gamma_0 \tag{2}$$

Obviously, the adjoint boundary value condition for (2) is

$$v = 0, \quad \text{on } \partial\mathscr{D} \tag{3}$$

Let

$$\widetilde{C}^2 \equiv \{u(x,y) \mid u \in C^2(\overline{\mathscr{D}}), u|_{\Gamma_0} = 0\}$$
$$\widetilde{C}^2_* \equiv \{v(x,y) \mid v \in C^2(\overline{\mathscr{D}}), v|_{\partial\mathscr{D}} = 0\}$$

On the coefficients of the equation (1) and Γ_0 we make the following assumptions

$$\begin{cases} \text{(i)} \quad (a - 2k_y)|_{y=0} \geqslant \delta_1 \\ \text{(ii)} \quad -(k_{yy} - a_y - \beta_x + 2\gamma) \geqslant \delta_2, \quad \text{in } \mathscr{D}^+ \\ \text{(iii)} \quad n_2|_{A,B} > 0, n_1|_{A,B} \neq 0, \quad \text{on } \Gamma_0 \end{cases} \tag{4}$$

where δ_1 and δ_2 are arbitrary positive constants, n_1 and n_2 are the components of the unit outer normal vector to the boundary $\partial\mathscr{D}$ respectively.

So far not many studies have been done on the non-linear equations of mixed type. In 1985, A. K. Aziz and M. Schneider issued a paper[1] on studying the non-linear equations of mixed type of first kind, and thus made an important step into the field of the non-linear equations of mixed type. Nevertheless, they only solved the problem of the existence of the generalized solution under the very restrictive conditions. As to the research on that respect, we shall publish another paper soon. In this paperr we shall deal with the modified Tricomi problem (1), (2) of the semi-linear mixed type equation of second kind. By using the energy-integeral method and the Leray-Schauder fixed point principle, we have established the existence and uniqueness of the $H_1(\mathscr{D})$ strong solution under the very weak condition (4).

1. Energy Inequality

Firstly, we consider the linear case:

$$\begin{aligned} L_0 u \equiv & k(x,y)u_{yy} + u_{xx} + a(x,y)u_y + \beta(x,y)u_x + \gamma(x,y)u \\ = & f(x,y) \end{aligned} \tag{5}$$

On the boundary value problem of the linear second-kind mixed type equation (5), many researches have been done (see [2]—[14]). [2] deals with the simplest case: $k = y, a = \text{const.}, \beta = \gamma = 0$; and [3] the case: $k = \operatorname{sgn} y |y|^m, 1 \leqslant m < 2, a = \tau |y|^{m-1}, m - 1 < \tau < 1, \beta = \gamma = 0$. In [4]—[7], the second-kind mixed type equations which are rela-