*J. Partial Differential Equations*
*Vol.* 3, *No.* 4(1990), 65—86

# QUALITATIVE STUDY OF A BOUNDARY VALUE PROBLEM FOR A REACTION-DIFFUSION SYSTEM①

Wu Yaping
(Dept. of Appl. Math., Beijing Institute of technology)
(Received June 10, 1988)

**Abstract** In this paper, the order of magnitude and structure of nonnegative solutions for a class of R-D systems with 3 components is obtained. Especially for the case $a_2 = a_3$, a complete analysis is given. By using different method from [1], a simpler and more concrete sufficient condition for uniqueness and globally asymptotic stability of nonnegative solution is also obtained.

**Key Words** nonnegative solutions; magnitude; reaction-diffusion system; uniqueness; globally asymptotic stability.

**Classification** 35K.

## 1. Introduction

In this paper, we investigate the steady-state solution of the following boundary value problem, which arises in the theory of chemical reactors.

$$\begin{cases} -b_1\Delta u(x) = v(x)w(x)g(u(x)) \\ -b_2\Delta v(x) = -v(x)w(x)g(u(x)) \qquad x\in\Omega \\ -b_3\Delta w(x) = -v(x)w(x)g(u(x)) \end{cases} \tag{1.1a}$$

$$\begin{cases} \partial_n u(x) + a_1 u(x) = 0 \\ \partial_n v(x) + a_2(v(x)-1) = 0 \qquad x\in\partial\Omega \\ \partial_n w(x) + a_3(w(x)-1) = 0 \end{cases} \tag{1.1b}$$

In the case $a_2 = a_3$, define $\lambda = b_3/b_2$, then every solution $(u(x), v(x), w(x))$ of (1.1a)(1.1b) satisfies

$$w(x) = 1 - 1/\lambda + v(x)/\lambda \tag{1.2}$$

Thus (1.1a) and (1.1b) become

$$\begin{cases} -b_1\Delta u(x) = v(x)(1-1/\lambda+v(x)/\lambda)g(u(x)) \\ -b_2\Delta v(x) = -v(x)(1-1/\lambda+v(x)/\lambda)g(u(x)) \end{cases} \qquad x\in\Omega \tag{1.3a}$$

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① Research supported by the National Natural Science Foundation of China.

$$\begin{cases}\partial_n u(x)+a_1 u(x)=0 \\ \partial_n v(x)+a_2(v(x)-1)=0\end{cases} \quad x\in\partial\Omega \qquad (1.3b)$$

$$w(x)=1-1/\lambda+v(x)/\lambda$$

The existence of nonnegative solutions of (1.1$a$) and (1.1$b$) has been proved in [1]. In [1], Xie Hong gives a sufficient condition for uniqueness and asymptotic stability of nonnegative solutions, he also gives a basic analysis of the structure of nonnegative solutions.

P. C. Fife, L. Hsiao, T. Zhang[2] have investigated the order of magnitude and structure of the solution to the following boundary value problem

$$\begin{cases}b\Delta u(x)=-v(x)g(u(x)) \\ b\Delta v(x)=v(x)g(u(x))\end{cases} \quad x\in\Omega$$

$$\begin{cases}\partial_n u(x)+a_1 u(x)=0 \\ \partial_n v(x)+a_2(v(x)-1)=0\end{cases} \quad x\in\partial\Omega$$

The estimates they obtained are sharp. The above boundary value problem describes a simple exothermic process of a reactant in chemical theory.

In this paper, similar estimates for problem (1.1$a$)(1.1$b$) are obtained, a priori estimates for the case $b_i\geqslant 1(i=1,2,3)$, and a complete analysis for the case $a_2=a_3$ are given. The main results in [2] are just the limiting cases in this paper. The method of our proof is to obtain a priori estimates for the solutions in the uniform and Hölder norms, and apply the Schauder Fixed Point Theorem. We also investigate the uniqueness and the asymptotic stability of the nonnegative solution by using a method different from [1]. The sufficient conditions we obtained is simpler and more concrete than that of [1].

## 2. Preliminary Lemmas and Main Results

**Lemma 2.1**[2] *Let $u(x)\in C^2(\Omega)\cap C^1(\overline{\Omega})$, $\varphi(x)$, $f(x)\in C(\overline{\Omega})$ satisfy*

$$\begin{cases}\Delta u(x)=f(x) & x\in\Omega \\ \partial_n u(x)+\varepsilon u(x)=\varphi(x) & x\in\partial\Omega=\Gamma\end{cases}$$

*where $\int_\Omega f(x)dx=\int_\Gamma \varphi(x)ds$, $\int_\Gamma u(x)ds=0$ and $\varepsilon\geqslant 0$. There is a constant $C_1$ depending only on $\Omega$, such that*

$$|u|_0\leqslant C_1(|f|_0+|\varphi|_0) \qquad (2.1)$$

**Lemma 2.2**[1] *Let $u(x)\in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfies*

$$\begin{cases}\Delta u(x)=f(x) & x\in\Omega \\ \varepsilon\partial_n u(x)+u(x)=0 & x\in\partial\Omega\end{cases}$$