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ON THE EXISTENCE AND REGULARITY OF NONTRIVIAL SOLUTIONS IN $W^{1,p}$ FOR QUASILINEAR ELLIPTIC EQUATIONS

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Many satisfactory results on the existence of nontrivial solutions in $W_0^{1,p}$ for quasilinear elliptic equations have been obtained by the Mountain Pass Lemma (see Shen Yaotian [2—4]). Recently, Wu Shaoping discussed the existence and regularity of the nontrivial solution in $W^{1,m}$ for the Neumann problem for a class of quasilinear equations. In paper [6], the author of this paper studied the existence of infinitely many critical points of the even functional $\int_\Omega F(x, u, Du) + \int_{\partial\Omega} G(x, u)$ in $W^{1,p}(\Omega)$. The work of this paper is the continuation of that in [6]. We study the existence of the nontrivial critical points, especially, the positive critical point and the negative critical point, of the functional (not necessarily even):

$$I(u) = \int_\Omega F(x, u, Du) + \int_{\partial\Omega} G(x, u), \quad u \in W^{1,p}(\Omega) \tag{1}$$

where $G(x, u) = \int_0^u g(x, t)\,dt$. For simplicity, we call the function u is positive if $u \geq 0$ but $u \not\equiv 0$ in Ω; and u is negative if $-u$ is positive. Furthermore, we discuss the regularity of the weak solutions of the second order quasilinear elliptic equations in divergence form under the weak structure conditions. Our regularity result generalizes the result in [9] which only deals with the semilineai elliptic equations.

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$, $n(x)$ be the unit outward normal to $\partial\Omega$ at x. Set $F_i(x, u, q) = F_{q_i}(x, u, q)$, $(i = 1, 2, \ldots, n)$. Formally, the critical point of the functional (1) is the weak solution of the following elliptic problem:

$$\begin{cases} -\dfrac{d}{dx_i} F_i(x, u, Du) + F_u(x, u, Du) = 0, & x \in \Omega \quad (2) \\ F_i(x, u, Du)\cos(n, x_i) + g(x, u) = 0, & x \in \partial\Omega \quad (3) \end{cases}$$

Set $\bar{p} = np/(n-p)$. Suppose $n \geq 3$ and $1 < p < n$. The norm for $W^{1,p}(\Omega)$ is $\|u\| = \left(\int_\Omega |Du|^p + \int_\Omega |u|^p \right)^{\frac{1}{p}}$. In the sequel, the contants C and C_ε depending on ε may vary from relation to relation in the argument.

For $m > 1$, set $m' = m/(m-1)$. First, we give the following conditions for $F(x, u, q)$ and $g(x, u)$:

(F_1) $F(x, 0, 0) = 0$, $(x \in \Omega)$; $F(x, u, q)$, $F_i(x, u, q)$ and $F_u(x, u, q) \in C(\bar{\Omega} \times R \times R^n)$, $(i = 1, 2, \dots, n)$;

(F_2) There is a $C > 0$ such that

$$|F_i(x, u, q)| \leq C|q|^{p-1} + \varphi_1(|u|)$$
$$|F_u(x, u, q)| \leq \varphi_2(|q|) + \varphi_3(|u|)$$

where $\varphi_i(t) \in C[0, +\infty)$, $(i = 1, 2, 3)$ and $\varphi_1(t) = o(t^{\bar{p}/p'})$, $\varphi_2(t) = o(t^{p/\bar{p}'})$, $\varphi_3(t) = o(t^{\bar{p}-1})$ as $t \to +\infty$;

(F_3) $(F_i(x, u, q) - F_i(x, u, \bar{q}))(q_i - \bar{q}_i) > 0$ $\quad (q \neq \bar{q})$;

(F_4) There exists $\lambda_1 > 0$ such that

$$F_i(x, u, q) q_i \geq \lambda_1 |q|^p - \varphi_4(|u|)$$

where $\varphi_4(t) \in C[0, +\infty)$ and $\varphi_4(t) = o(t^{\bar{p}})$ as $t \to +\infty$;

(F_5) There are $0 < \theta < \dfrac{1}{p}$, $\beta > 0$, $k > 0$, such that

$$F(x, u, q) - \theta(F_i(x, u, q) q_i + F_u(x, u, q) u) \geq \beta |q|^p + \beta |u|^p - k$$

(F_5') There are $0 < \theta < \dfrac{1}{p}$, $\beta > 0$, $k > 0$, such that

$$F(x, u, q) - \theta(F_i(x, u, q) q_i + F_u(x, u, q) u) \geq \beta |q|^p - k$$

(F_6) $F(x, u, q) \leq C|q|^p - \Phi(|u|)$, where $C > 0$, $\Phi(t) \in C[0, +\infty)$ and $\lim_{t \to +\infty} \Phi(t)/t^p = +\infty$;

(F_6') For all $u \geq 0$, $F(x, u, q) \leq C|q|^p - \Phi(u)$, where $C > 0$, $\Phi(t) \in C[0, +\infty)$ and $\lim_{t \to +\infty} \Phi(t)/t^p = +\infty$.

(g_1) $g(x, u) \in C(\bar{\Omega} \times R)$, $g(x, 0) = 0$ $\quad (x \in \Omega)$;

(g_2) There is a $r > 0$ such that for $|u| > r$,

$$pG(x, u) \geq ug(x, u) \geq 0$$

(g_2') There exists $r > 1$ such that for $|u| > r$,

$$pG(x, u) \geq ug(x, u) \geq a(x)|u|$$

where $a(x)$ is a nonnegative continuous function on $\bar{\Omega}$, and $\int_{\partial\Omega} a(x) > 0$.

Theorem 1 *Suppose that $F(x, u, q)$ satisfies (F_1)—(F_4), (F_5'), (F_6') and*

(F_7) *There is a $v > 0$ such that*

$$F(x, u, q) \geq v|q|^p - P(x, u)$$

where $P(x, u) \in C(\bar{\Omega} \times R)$, $|P(x, u)| \leq a_1 |u|^p + a_2$ and $P(x, u) = o(|u|^p)$ uniformly in $x \in \Omega$ as $u \to 0$.