J. Partial Differential Equations
Vol. 2, No. 2 (1989), 87—96

GENERALIZED TRICOMI PROBLEM FOR THE NONLINEAR EQUATION OF MIXED TYPE①

Sun Hesheng Xiao Zhuang
(Institute of Applied Physics and Computational Mathematics)
(Received May 3, 1988)

1. Introduction

In domain $D\ (=D^+\cup D^-)$ we consider the nonlinear equation of mixed type

$$Lu\equiv k(x,\ y)u_{xx}+u_{yy}+\alpha(x,\ y)u_x+\beta(x,\ y)u_y+\gamma(x,\ y)u-|u|^\rho u=f(x,\ y) \tag{1}$$

where the function $k(x,y)$ satisfies the conditions: $yk>0$, when $y\neq 0$, $k(x,0)=0$, $k\in C^1(\bar{D})$, the functions $\alpha,\beta,\gamma\in C(\bar{D})$, $f\in L_2(D)$, $\rho>0$ is a constant. Let the outer boundary of D^+ be an arbitrary piecewise smooth curve Γ_0, which is connected with the degenerating line $y=0$ on points A and B. Let the outer boundary of D^- be two families of characteristic curves Γ_+ and Γ_-, issuing from the corresponding points A and B respectively and defined by the equations $dx+\sqrt{-k}dy=0$ and $dx-\sqrt{-k}dy=0$ respectively. Let Γ'_+ be an arbitrary peecewise smooth curve, issuing from the intersection point A of Γ_+ and Γ_0, and lying inside the characteristic triangle, but the slope of the curve Γ'_+ is not less than the slope of the characteristic line of the family Γ_+ on the corresponding points, and Γ'_+ is not tangent to the family Γ_-. Γ'_+ is defined by the following equation:

$$\Gamma'_+:\ dx+\xi(x,\ y)dy=0,\ \xi\geq\sqrt{-k}\quad\text{on}\quad\Gamma'_+ \tag{2}$$

For Eg. (1) we consider the following generalized Tricomi problem (or Tricomi problem):

$$(L-\lambda)u=f \quad \text{in} \quad D \tag{3}$$

$$u=0 \quad \text{on} \quad \Gamma_0\cup\Gamma'_+\ (\text{or}\quad \Gamma_0\cup\Gamma_+) \tag{4}$$

where λ is a positive constant.

For the linear case when $\rho=0$, the problem (3) (4) is considered by us in [1],

①Project supported by the National Natural Science Foundation of China

where one condition on Γ_0 is needed, and this condition is weakened by applying the Schwarz's alternating procedure analogous to that used in [2, 3]. But applying this procedure is more complicated. In this paper by means of the method of energy integral we succeed in choosing a set of functions a, b and c, and get the same elegant result as [1] and [2] without applying the Schwarz's alternating procedure, so we simplify the works [1] and [2]. Moreover, we solve the problem (3) (4) for the nonlinear equation (1). Under quite strong conditions on coefficients of equation and on curve Γ_0, only the weak solution of problem (1) (4) is obtained in [4] (also [5]) by applying the Galerkin's method. In this paper we consider the nonlinear problem (3) (4), under quite weak conditions, we prove the existence and uniqueness of strong solution for the problem (3) (4) by applying the energy estimations and the fixed point principle.

2. Linear Problem

In order to solve the nonlinear problem (3) (4), firstly we consider the linear equation

$$(L_0 - \lambda)u \equiv ku_{xx} + u_{yy} + \alpha u_x + \beta u_y + \gamma u - \lambda u = f \tag{5}$$

For the equation (5) we place restrictions on the cure Γ_0 and the coefficients $k(x, y)$ and $\alpha(x, y)$ as follows:

$$\begin{cases} (i)\ \textit{There is a positive constant } \delta, \\ \quad (\xi^{-1}k_y - 2\alpha)|_{A^-} \geq \delta \quad \text{if} \quad k_y|_{y=0} > 0 \quad \text{and} \quad \xi|_{A^-} > 0, \\ \quad \text{or} \quad -\alpha(x, 0) \geq \delta \quad \text{if} \quad k_y|_{y=0} = 0; \\ (ii)\ n_2|_{B^+} > 0 \quad \text{and} \quad n_2|_{A^+} > 0 \quad \text{on} \quad \Gamma_0, \\ \quad \text{or} \quad \Gamma_0 \quad \textit{connects} \quad \Gamma'_+ (\Gamma_+) \text{ on point } A \text{ smoothly and } (\xi n_2 - n_1)|_{A^-} \geq 0 \end{cases} \tag{6}$$

where n_1 and n_2 are the components of the unit vector of the cutward normal to the boundary curves. $A^+ (A^-)$ denotes the limiting point in $y \geq 0$ $(y \leq 0)$.

Remark 1 If $\xi|_{A^-} = 0$ and $k_y|_{y=0} > 0$, then the condition (6) (i) is not needed. Hence, for the Tricomi problem (i. e. $\xi \equiv \sqrt{-k}$) or the generalized Tricomi problem in the case when a section of Γ'_+ coincides with Γ_+, the restriction on coefficients is not needed any more unless $k_y|_{y=0} > 0$.

Make the double integral