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COMMENT ON SOME ALGEBRAIC PROPERTIES OF THE GAUGE EQUIVALENT SOLITON EQUATIONS①

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Abstract

In references to [1], [2], a theory is proposed for generating the constants of motion, the symmetries and their Lie algebra for some dynamical systems, where three parameters α, β, h are introduced. In this paper, we have proved that if the soliton equations in dynamical systems are gauge equivalent, then the parameters α, β, h are invariant and the Lie algebra of the symmetries is isomophic under the transformation operator. At last some interesting examples are given.

1. Introduction

A soliton equation

$$u_t = \Phi K_0 = K(u) \tag{1.1}$$

where $u=u(x, t)$ may be vector function with n components which belong to the Schwartz space. $K_0=K_0(u)$ is some differentiable map on this space depending on u and on derivatives of u with respect to x, and $\Phi=\Phi(u)$ is a linear integro-differential operator called recursion operator, strong symmetry or hereditary operator[3][4], and Φ can be factorized by implectic operator,

$$\Phi = \theta J \quad , \quad J = \theta^{-1} \tag{1.2}$$

The equation (1.1) can be deduced from the Lax equation

$$L_t = ML - LM \tag{1.3}$$

where

$$L\varphi = 0, \qquad \varphi_t = M\varphi \tag{1.4}$$

and $L=L(u, \lambda)$ is a differential operator, $M=M(u, \lambda)$ is an operator, λ is a parameter and $\varphi=\varphi(x, t, \lambda)$ is a vector function.

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By the gauge transformation

$$\delta = G\varphi, \quad G = G(u, s, \lambda) \tag{1.5}$$

(where $\overline{\varphi}=\overline{\varphi}(x, t, \lambda)$ and $s=s(x, t)$ are vector functions and s belongs to the Schwartz space.) $\overline{\varphi}$ satisfies the equations

$$\overline{L}\overline{\varphi} = 0, \quad \overline{\varphi}_t = \overline{M}\overline{\varphi} \tag{1.6}$$

where $\overline{L}=\overline{L}(s, \lambda)$, $\overline{M}=\overline{M}(s, \lambda)$; they are connected with L, M by

$$\overline{L}G - GL = 0, \quad G_t = \overline{M}G - GM \tag{1.7}$$

and the vector functions u and s are connected by the Backlund transformation

$$B(s, u) = 0 \tag{1.8}$$

If the operator $T=B_s^{-1}B_u$ is invertible[3],[4], then we set

$$\overline{K}_0(s) = -TK_0(u) \tag{1.9}$$

$$\overline{\Phi}(s) = T\Phi(u)T^{-1} \tag{1.10}$$

the operator T maps the equation (1.1) to the equation

$$s_t = \overline{\Phi}\overline{K}_0 = \overline{K}(s) \tag{1.11}$$

The operator T is called transformation operator and the equation (1.11) is called the gauge equivalent equation with the equation (1.1).

It is well known that one important problem in matrix theory is for finding the invariants of the matrix under the similarity transformation. We notice the similarity between the soliton theory and the matrix theory, the operator Φ and T play the role of the matrix and the similarity transformation respectively in matrix theory. As we know many remarkable algebraic properties of the soliton equations have been found recently; such that to find the invariants of the equations under the transformation operator is interesting in soliton theory.

In references [1], [2], we assume that $\sigma_0=\sigma_0(u)$ or $\sigma_1=\sigma_1(u)$ is a field which associates with the Galilean invariant or scaling invariant of the equation (1.1), then three parameters α, β, h are introduced as follows:

1) α: $\quad \Phi'[\sigma_i]=\sigma_i'\Phi-\Phi\sigma_i'+\alpha\Phi^i, \ i=0, 1$ (1.12)

2) β: $\quad [K_0, \sigma_0]=0, \ [K_0, \sigma_1]=\beta K_0$

or $[K_1, \sigma_1]=(\alpha+\beta)K_1, \ K_1=\Phi K_0$ (1.13)

3) h: $\quad J'[\sigma_1]+J\sigma_1'+\sigma_1'^{*}J=\alpha hJ$ (1.14)

For the equations

$$u_t = K_l = \Phi^l K_0 \tag{1.15}$$

there are two sets of symmetries

$$K_m = \Phi^m K_0, \quad \tau_n^l = \Phi^n \tau_0^l, \ \tau_0^l = (\alpha l + \beta)tK_{l-1} + \sigma_0 \tag{1.16}$$

A graded Lie algebra is valid for these K_m and τ_n^l-symmetries

$$[K_m, K_n] = 0$$