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A NOVEL FREE BOUNDARY PROBLEM—THE SAUNA[①]

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Abstract

We consider a novel free boundary problem-the Sauna, and prove the existence and uniqueness of solution for its weak form.

1. Introduction

A variety of free boundary problems arise in the nature and industry. And certain phenomena in daily life can be reduced to free boundary problems, too. For example, when people wearing glasses enter a heated room from outside on a cold day, their glasses become steamed up. Then if the glasses are not wiped, after some time, the water on the lenses starts to evaporate. A similar situation occurs in a Sauna (See [1]).

In [1], the authors presented an interesting mathematical model concerning these phenomena and the numerical results.

In this paper, we analyze the Sauna model in theory. In Section 2, we formulate this problem in weak form, then prove the existence and uniqueness theorems for weak solutions in Section 3, and investigate some properties of weak solutions in Section 4.

2. Weak Formulation

As in [1], if we only consider axially symmetric situations, the mathematical model for the Sauna is as follows[②]:

$$\alpha_g \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \theta}{\partial x}\right) + 1 - \theta + \frac{\partial z}{\partial \tau} = \frac{\partial \theta}{\partial \tau}, \quad \text{in} \quad 0 \leq x \leq 1 \qquad (2.1)$$

$$\alpha_f \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \theta}{\partial x}\right) + 1 - \theta + \frac{\partial z}{\partial \tau} = \gamma\frac{\partial \theta}{\partial \tau}, \quad \text{in} \quad 1 \leq x \leq 1 + c \qquad (2.2)$$

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② Throughout this paper, the meaning of following notation and constants is the same as that in [1].

$$\theta(1-,\tau)=\theta(1+,\tau) \quad \text{and} \quad \alpha_s\frac{\partial\theta}{\partial x}(1-,\tau)=\alpha_f\frac{\partial\theta}{\partial x}(1+,\tau) \tag{2.3}$$

$$\frac{\partial z}{\partial\tau}=\beta(1-\mu(\theta)) \tag{2.4a}$$

if $z>0$ or $z=0$ and $u(\theta)<1$, and

$$\frac{\partial z}{\partial\tau}=0 \tag{2.4b}$$

if $z=0$ and $u(\theta)>1$.

$$\theta(x,0)=\theta_0(x) \quad \text{in} \quad 0<x<1+c \tag{2.5}$$

$$\frac{\partial\theta}{\partial x}(1+c,\tau)=0 \tag{2.6}$$

$$z(x,0)=0 \tag{2.7}$$

(2.1)—(2.7) are the Sauna problem. The boundary of set $\{(x,y,t)\,|\,z(x,y,t)=0\}$ is the free boundary.

In our case, we consider non-axially symmetric situation and introduce Heaviside function:

$$H(\xi)=\begin{cases}0 & \xi\le 0\\ 1 & \xi>0\end{cases} \tag{2.8}$$

Setting

$$h(\theta)=\beta\ \max(0,1-u(\theta))=\beta(1-u(\theta))^+ \tag{2.9a}$$

and

$$g(\theta)=\beta\ \min(0,1-u(\theta))=-\beta(1-u(\theta))^- \tag{2.9b}$$

we get the following equation

$$\frac{\partial z}{\partial\tau}=h(\theta)+g(\theta)H(z) \tag{2.10}$$

equivalent to (2.4a) and (2.4b).

So, we obtain the general Sauna problem as follows:

$$\alpha_s(\theta_{xx}+\theta_{yy})+1-\theta+z_\tau=\theta_\tau \qquad \text{in} \quad Q_1 \tag{2.11}$$

$$\alpha_f(\theta_{xx}+\theta_{yy})+1-\theta+z_\tau=\gamma\theta_\tau \qquad \text{in} \quad Q_2 \tag{2.12}$$

$$\left.\begin{aligned}&\theta(x,y,\tau)|_{r=1-}=\theta(x,y,\tau)|_{r=1+}\\ &\alpha_s\frac{\partial\theta}{\partial n}\Big|_{r=1-}=\alpha_f\frac{\partial\theta}{\partial n}\Big|_{r=1+}\end{aligned}\right\} \tag{2.13}$$

$$\frac{\partial\theta}{\partial n}=0 \qquad \text{on} \quad r=1+c \tag{2.14}$$

$$\theta(x,y,0)=\theta_0(x,y) \qquad \text{in} \quad \Omega \tag{2.15}$$

$$z(x,y,0)=0 \qquad \text{in} \quad \Omega \tag{2.16}$$

(2.10)——(2.16) are called problem P1, where

$$\Omega=\left\{(x,y)\,|\,\sqrt{x^2+y^2}=r<1+c\right\}, \quad Q=\Omega\times(0,T)$$