J. Partial Differential Equations
Vol. 1 № 1 (1988), 31—41

CAUCHY PROBLEM FOR A CLASS OF TOTALLY CHARACTERISTIC HYPERBOLIC OPERATORS WITH CHARACTERISTICS OF VARIABLE MULTIPLICITY IN GEVREY CLASSES①

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Received November 17, 1986; revised June 25, 1987

Abstract

This paper studies the Cauchy problem of totally characteristic hyperbolic operator (1.1) in Gevrey classes, and obtains the following main result:

Under the conditions (I) — (VI), if $1\leqslant s<\frac{\sigma}{\sigma-1}$ (σ is defined by (1·7)), then the Cauchy problem (1·1) is wellposed in $B([O, T], G^s_{L^2}(R^n))$; if $s=\frac{\sigma}{\sigma-1}$, then the Cauchy problem (1.1) is wellposed in $B([0, \varepsilon], G^{\frac{\sigma}{\sigma-1}}_{L^2}(R^n))$ (where $\varepsilon>0$, small enough).

1. Main Result

In this paper, we consider the Cauchy problem of totally characteristic hyperbolic operator with weight m-k in t, i.e.

$$\begin{cases} Pu=(t^kD_t^m+P_1(t,x;D_x)t^{k-1}D_t^{m-1}+\cdots+P_k(t,x;D_x)D_t^{m-k}+\cdots \\ \quad +P_m(t,x;D_x))u(t,x)=f(t,x),\quad (t,x)\in\Omega=[0,T]\times R^n \\ D_t^ju(t,x)|_{t=0}=u_j(x),\qquad 0\leqslant j\leqslant m-k-1 \end{cases} \tag{1·1}$$

Problem (1·1) was discussed by [1], [2]; but in this paper our conditions are different from those in [1] or [2]. Suppose

(I). $k\in \mathbf{Z}_+$, $\quad 0\leqslant k\leqslant m$

(II). Order $P_j(t,x;D_x)\leqslant j$, $\quad 1\leqslant j\leqslant m$

(III). $P_j(t,x;D_x)=\sum_{|\beta|\leqslant j}a_{j\beta}(t,x)D_x^\beta$,

$$a_{j\beta}(t,x)\in B([0,T],G^s(R^n))\qquad (s\geqslant 1,\ 1\leqslant j\leqslant m)$$

(IV). The characteristic polynomial of P satisfies

$$\tau^m+\sum_{j=1}^m[t^{\max(0,j-k)}\cdot\sum_{|\beta|=j}a_{j\beta}(t,x)\xi^\beta]\tau^{m-j} \tag{1·2}$$
$$=\prod_{i=1}^{m_1}(\tau-\lambda_i(t,x;\xi))\cdot\prod_{j=1}^{m_2}(\tau-t^q\mu_j(t,x;\xi))$$

where $m_1+m_2=m$, $m_2\geqslant 2$; $q>0$, a rational number; $\lambda_i(t,x;\xi)$, $\mu_j(t,x;\xi)\in B([0,T],S^1_{\rho,\delta})$ are all real valued functions on $\Omega\times R^n$; if $(t,x)\in\Omega$, $|\xi|=1$, we have: $\lambda_i(t,x;\xi)\neq\lambda_j(t,x;\xi)$ $(1\leqslant i\neq j\leqslant m_1)$, $\mu_i(t,x;\xi)\neq\mu_j(t,x;\xi)$ $(1\leqslant i\neq j\leqslant m_2)$ and $\lambda_i(0,x;\xi)\neq 0$ $(1\leqslant i\leqslant m_1)$.

The indicial operator of P

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① Supported by the Science Foundation of Academia Sinica

$$L(\lambda, x; D_x) = \lambda(\lambda-1)\cdots(\lambda-m+1) + P_1(0, x; D_x)\lambda(\lambda-1)\cdots(\lambda-m+2) + \cdots + P_k(0, x; D_x)\lambda(\lambda-1)\cdots(\lambda-m+k+1)$$

is a non-singular operator of order k with parameter λ, we assume

(V). $L(\lambda, x; D_x)$ is uniquely solvable in $G^s_{L^2}(R^n)$ for any $\lambda \in \mathbf{Z}$, such that $\lambda \geqslant m-k$.

Under the conditions above, Tahara [3] considered the H^∞ wellposed of Cauchy problem (1 · 1), but in [3], the lower order part of operator (1 · 1) was restricted. In this paper, in order to solve the problem (1. 1) and improve the result of [3], we use successive approximation method in Gevrey classes, thus the restrictions in lower order terms of operator (1. 1) are weakened. Let

$$P = \bar{P} + \bar{\bar{P}} \tag{1 · 3}$$

$\bar{P} = t^k D_t^m + \sum_{j=1}^{m} \sum_{|\beta|=j} a_{j\beta}(t, x)\, t^{\max(0, k-j)} D_t^{m-j} D_x^\beta + \sum_{j=1}^{m} a_{j0}(t, x)\, t^{\max(0, k-j)} D_t^{m-j}$ is the principal part of P; $\bar{\bar{P}} = \sum_{j=2}^{m} \sum_{1 \leqslant |\beta| \leqslant j-1} a_{j\beta}(t, x)\, t^{\max(0, k-j)} D_t^{m-j} D_x^\beta$ is the lower order part of P.

Using successive approximation method we can get the formal solution series of Cauchy problem (1. 1). Thus we have to impose some restrictions on coefficients of $\bar{\bar{P}}$ in order to ensure the convergence of the formal solution series, namely

(VI). $a_{j\beta}(t, x) = t^{w(j, \beta)} \hat{a}_{j\beta}(t, x)$, $\quad 1 \leqslant |\beta| \leqslant j-1, \quad 2 \leqslant j \leqslant m.$

where $w(j, \beta) \in \mathbf{Z}_+$, and $w(j, \beta) = 1$ if $1 \leqslant |\beta| \leqslant j-1$, $2 \leqslant j \leqslant k$.

In [9], the index of G^s-wellposed was introduced. Here we will see the index of G^s-wellposed of operator (1, 1) depends on the order of degeneracy of principal part and the coefficients of lower order terms of operator (1, 1). Set

$$d(m-j+|\beta|, \beta) = \begin{cases} w(j, \beta), & 1 \leqslant |\beta| \leqslant j-1, \ 2 \leqslant j \leqslant k \\ w(j, \beta) + j - k, & 1 \leqslant |\beta| \leqslant j-1, \ k+1 \leqslant j \leqslant m \end{cases} \tag{1 · 4}$$

then $d(m-j+|\beta|, \beta) \geqslant 1$ is a positive integer. Define

$$\sigma_i = \max_{1 \leqslant |\beta| \leqslant i} \left(|\beta| - \frac{d(i, \beta)}{q}; 0\right), \quad (1 \leqslant i \leqslant m-1) \tag{1 · 5}$$

and for any positive integers $k_i \geqslant \sigma_i$, $(1 \leqslant k_i \leqslant m-1, 1 \leqslant i \leqslant m-1)$, suppose

$$\gamma = \max_{1 \leqslant i \leqslant m-1} \left(\frac{\sigma_i}{k_i}\right), \quad (\in [0, 1]) \tag{1 · 6}$$

$$\sigma = \max_{1 \leqslant i \leqslant m-1} \left(\frac{k_i \gamma + m - i}{m-i}\right) \tag{1 · 7}$$

then $\sigma \geqslant 1$, and $\frac{\sigma-1}{\sigma}$ is the index of G^s-wellposed of operator (1 · 1).

Our main result is as follows:

Theorem A: *Under the conditions* (I—VI), *for any* $u_j(x) \in G^s_{L^2}(R^n)$ $(0 \leqslant j \leqslant m-k-1)$ *and* $f(t, x) \in B([0, T], G^s_{L^2}(R^n))$, *if* $1 \leqslant s < \frac{\sigma}{\sigma-1}$, *the Cauchy problem* (1 · 1) *has a unique solution in* $B([0, T], G^s_{L^2}(R^n))$; *if* $s = \frac{\sigma}{\sigma-1}$, *then the Cauchy problem* (1 · 1) *has a unique local solution in* t $u(t, x) \in B([0, \varepsilon], G^{\frac{\sigma}{\sigma-1}}_{L^2}(R^n))$ *(where* $\varepsilon > 0$, *small enough).*

Similar to [1] and [2], by Borel's technique, theorem A can be deduced from the following result: