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GLOBAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS AND CAUCHY PROBLEM FOR TWO TYPES OF NONLINEAR PSEUDO-PARABOLIC SYSTEMS *

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1. Introduction

Various types of nonlinear evolutional systems of partial differential equations can be derived from a great amount of modern research in physics, chemical reactions, mechanics etc. such as the linear or nonlinear parabolic or pseudo-parabolic systems. So that it is meaningful and interesting to discuss the well-posedness in global of some basic problems as periodic boundary problems, initial-boundary value problems and Cauchy problems for the above mentioned systems. There appeared some papers concerned about the problems of some types of the nonlinear pseudo-parabolic equations and systems[1-12]. In this paper we are going to consider some problems for two new types of the nonlinear pseudo-parabolic systems, which are, in particular, different from the systems discussed in [1, 2].

Now let us give some conventions of notations for the following use.

Denote by (u, v) the inner product of two vectors u and v, $(u, v) = \int_{-X}^{X} u \cdot v dx$, also $|u(\cdot, t)|^2_{L_2(\Omega)} = (u, u)$, $\Omega \equiv (-X, X)$.

Denote by $[u, v]$ the integration from 0 to t of the inner product of two vectors u and v,

$$[u, v] = \int_0^t (u, v) dt = \iint_{Q_t} u \cdot v dx dt$$

also $\| u \|^2_{L_2(Q_t)} = [u, u]$, $Q_t = \{ (x, \tau) | x \in \Omega \equiv (-X, X), \tau \in (0, t) \}$. Denote by $L_2(0, T; H^m(\Omega))$ the collection of functions (or vectors) $u(x, t)$, which when regarded as functions (or vectors) of variable x belong to space $H^m(\Omega)$ and when whose norms $|u(\cdot, t)|_{H^m(\Omega)}$ are regarded as functions (or vectors) of variable t belong to the space $L_2(0, T)$.

The following two lemmas will be used repeatedly in this paper.

Lemma 1 *(Nirenberg's lemma)*[13]. *If* $\int_{-X}^{X} v dx = 0$ *or* $v|_{x=-X} = v|_{x=X} = 0$, *then we have*

$$|v(\cdot, t)|_{L_p(\Omega)} \leq C |v_x(\cdot, t)|^{a}_{L_q(\Omega)} |v(\cdot, t)|^{1-a}_{L_r(\Omega)}, \ \forall t \in [0, T] \quad (1)$$

where $\frac{1}{p} = a(\frac{1}{q} - 1) + (1-a)\frac{1}{r}$, $a \in [0, 1]$, $r \geq 1$, $1 < p, q \leq \infty$.

In particular, we have

$$|v(\cdot, t)|_{L_\infty(\Omega)} \leq C |v_x(\cdot, t)|^{\frac{1}{2}}_{L_2(\Omega)} |v(\cdot, t)|^{\frac{1}{2}}_{L_2(\Omega)}, \ \forall t \in [0, T] \quad (2)$$

Lemma 2[14]. *Let* $Q_T = \{ (x, t) | x \in \Omega, t \in (0, T) \}$. *Suppose that* $G(Z_1, Z_2, \dots, Z_s)$

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is a function of g vectors $Z_1, Z_2, \dots, Z_g$, k (≥ 1) *— times continuously differentiable, and* $Z_i(x,t) \in L_\infty(Q_T) \cap L_2(0,T;H^k(\Omega))$. *Let* $\overline{M} = \max_{1\leq i\leq g} \sup_{(x,t)\in Q_T} |Z_i(x,t)|$. *Then, we have the inequality*

$$\left|\frac{\partial^k}{\partial x^k} G(Z_1(\cdot,t), \dots, Z_g(\cdot,t))\right|^2_{L_2(\Omega)} \leq C(\overline{M}, k, g) \sum_{i=1}^{g} |Z_i(\cdot,t)|^2_{H^k(\Omega)},$$
$$\forall t \in [0, t] \tag{3}$$

2. Periodic Boundary Problem. Cauchy Problem

In a rectangular domain $Q_T = \{(x,t) \mid x \in \Omega \equiv (-X, X), t \in (0,T)\}$ we consider the nonlinear pseudo-parabolic system

$$Lu \equiv u_t + (-1)^M A(t) u_{x^{2M}} + (-1)^M B u_{x^{2M}t} + \sum_{j=0}^{M} (-1)^j \frac{\partial^j}{\partial x^j}(F_j(u, \dots, u_{x^{M-1}}) u_{x^j}) = f(x,t) \tag{4}$$

and the periodic boundary problem

$$\begin{cases} u(x,t) = u(x+2X, t) \\ u(x,0) = \varphi(x) \end{cases} \tag{5}$$

where $M \geq 1$ is an integer, $u = (u_1, \dots, u_N)^T$ and $f = (f_1, \dots, f_N)^T$ are N-dimensional vector valued functions, $A(t)$ and B are $N \times N$ symmetric matrices, $F_j(u, \dots, u_{x^{M-1}})$ $(j = 0, 1, \dots, M)$ are arbitrary nonlinear functions of N-dimensional vector variables $p_m = (p_{1m}, \dots, p_{Nm})$, $p_{km} = u_{kx^m}$ $(m = 0, 1, \dots, M-1; k = 1, \dots, N)$.

Assume that the system (4) satisfies the following conditions:

$$\begin{cases} i) \ A(t) \text{ and } A'(t) \text{ are bounded matrices,} \\ ii) \ B \text{ is a positively definite constant matrix:} \\ \quad (B\xi, \xi) \geq b_0(\xi, \xi), \ \forall \xi \in R^N, \ b_0 > 0 \\ iii) \ F_j \ (j = 0, 1, \dots, M) \text{ are nonnegative and } m+1\text{-times} \\ \quad \text{continuously differentiable, } m \geq M, \\ iv) \ f_{x^{m-M}t} \in L_2(Q_T), \ \varphi(x) \in H^{M+m}(\Omega), \ m \geq M, \\ \quad f_k \text{ and } \varphi_k \text{ are periodic functions of } x \text{ with period} \\ \quad 2X, \ k = 1, \dots, N, \ f = (f_1, \dots, f_N), \ \varphi = (\varphi_1, \dots, \varphi_N) \end{cases} \tag{6}$$

Lemma 3. *The solutions of problem (4) (5) satisfy the following estimation*

$$|u(\cdot,t)|_{H^M(\Omega)} \leq C\{\|f\|^2_{L_2(Q_T)} + |\varphi|^2_{H^M(\Omega)}\}, \ \forall t \in [0, T] \tag{7}$$

Proof. Taking the scalar product of the vector u and the system (4) and then making the integration $[Lu, u]$ in the domain Q_t $(0 < t \leq T)$, we get

$$\frac{1}{2}(u,u) + \frac{1}{2}(Bu_{x^M}, u_{x^M}) + \sum_{j=0}^{M} [F_j(u, \dots, u_{x^{M-1}}) u_{x^j}, u_{x^j}]$$
$$= [f, u] + \frac{1}{2}(\varphi, \varphi) + \frac{1}{2}(B\varphi^{(M)}, \varphi^{(M)}) - [A(t) u_{x^M}, u_{x^M}] \tag{8}$$

Since $F_j \geq 0$ and the matrix $A(t)$ is bounded, then by applying the Gronwall's lemma we obtain the estimation (7)

Lemma 4. *The solutions of problem (4) (5) satisfy the estimate*

$$|u(\cdot,t)|_{H^{M+m}(\Omega)} \leq C(b_0, \|D_x^{m-M} f\|_{L_2(Q_T)}, |\varphi|_{H^{M+m}(\Omega)}), \ \forall t \in [0, T], \ m \geq M \tag{9}$$

Proof. By making the integration $[Lu, (-1)^m u_{x^{2m}}]$ $(m \geq 1)$, we have

$$\frac{1}{2}(u_{x^m}, u_{x^m}) + \frac{1}{2}(Bu_{x^{M+m}}, u_{x^{M+m}}) + \sum_{j=0}^{M}\left[\frac{\partial^m}{\partial x^m}(F_j(u, \dots, u_{x^{M-1}}) u_{x^j}), u_{x^{m+j}}\right]$$
$$= [f, (-1)^m u_{x^{2m}}] + \frac{1}{2}(\varphi^{(m)}, \varphi^{(m)})$$