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THE UNIQUENESS OF VISCOSITY SOLUTIONS OF THE SECOND ORDER FULLY NONLINEAR ELLIPTIC EQUATIONS

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Abstract

Recently R. Jensen [1] has proved the uniqueness of viscosity solutions in $W^{1,\infty}$ of second order fully nonlinear elliptic equation $F(D^2u, Du, u)=0$. He does not assume F to be convex. In this paper we extend his result [1] to the case that F can be dependent on x, i. e. prove that the viscosity solutions in $W^{1,\infty}$ of the second order fully nonlinear elliptic equation $F(D^2u, Du, u, x)=0$ are unique. We do not assume F to be convex either.

1. Introduction

This paper deals with the problem of uniqueness of viscosity solutions of the fully nonlinear second order elliptic partial differential equation

$$F(D^2u, Du, u, x)=0 \qquad \text{in} \quad \Omega \tag{1.1}$$

with Dirichlet boundary condition

$$u=g \qquad \text{on} \quad \partial\Omega \tag{1.2}$$

For any $\varepsilon>0$ we define

$$F_\varepsilon^+(D^2u, Du, u, x)=F(D^2u, Du, u, x+\varepsilon Du/(1+|Du|^2)^{\frac{1}{2}}) \quad \text{in} \quad \Omega_\varepsilon \tag{1.3}$$

$$F_\varepsilon^-(D^2u, Du, u, x)=F(D^2u, Du, u, x-\varepsilon Du/(1+|Du|^2)^{\frac{1}{2}}) \quad \text{in} \quad \Omega_\varepsilon \tag{1.4}$$

where $\Omega_\varepsilon=\{x\in\Omega\,|\,\mathrm{dist}(x,\partial\Omega)>\varepsilon\}$

In 1983 the definition of "viscosity solution" was introduced by M. G. Crandall and P. L. Lions [2] as a notion of weak solution of Hamilton-Jacobi equation

$$H(Du, u, x)=0 \qquad \text{in} \quad \Omega \tag{1.5}$$

Under some assumptions, they have established global uniqueness and existence of viscosity solutions. In P. L. Lions work[4] the definiton of "viscosity solution" was extended to second order problems, i. e., to (1.1), and under some regularity assumptions on F which include the convexity of F, the uniqueness of viscosity solutions was proved. Finally R. Jensen[1] proved uniqueness of viscosity solutions of (1.1) and (1.2) in 1986. He does not assume F to be convex but only not allow spatial dependence in x. The techniques he used in [1] are new. He constructed two approximation operators $A_\varepsilon^+[u]=u_\varepsilon^+\geq A_\varepsilon^-[u]=u_\varepsilon^-$ and proved his result.

In this paper we prove a maximum principle of viscosity solutions which implies the uniqueness of viscosity solutions of (1.1) and (1.2) in the two cases: (α) F is degenerate elliptic, decreasing and uniformly continuous in x; or (β) F is uniformly elliptic, nonincreasing, Lipschitz continuous in p and uniformly continuous in x. We do not assume F to be convex either. The techniques which we use are similar to that in [1] but with some improvement. First we prove that $A_\varepsilon^+[\cdot]$ takes viscosity subsolutions of (1.1) into viscosity subsolutions of $F_\varepsilon^+[\cdot]=0$ and $A_\varepsilon^-[\cdot]$ takes viscosity supersolutions into viscosity supersolutions of $F_\varepsilon^-[\cdot]=0$. Then we obtain an estimation of semiconvex functions. Lastly we combine these results with results of [1] and give the maximum principle of viscosity solutions.

We implicitly assume throughout this paper that Ω is a bounded domain in $\mathbf{R}^n$, g is continuous on $\partial\Omega$ and solutions of (1.1) and (1.2) are always in $C(\bar{\Omega})$.

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2. Viscosity Solutions

We begin by recalling some definitions. The set of $n \times n$ real symmetric matrices will be denoted by $S(n)$. These matrices admit the partial ordering $\succ$ where $M \succ N$ if $M - N$ is positive semidefinite. A fully nonlinear P. D. O. $F[\cdot]$ is defined by

$$F[\varphi](x) = F(D^2\varphi, D\varphi, \varphi, \cdot)(x) \quad \text{for all} \quad \varphi \in C^\infty(\Omega) \tag{2.1}$$

where $F \in C(S(n) \times \mathbf{R}^n \times \mathbf{R} \times \Omega)$.

Definition 2.1. *The operator $F[\cdot]$ is degenerate elliptic if*

$$F(M, p, t, x) \geq F(N, p, t, x) \tag{2.2}$$

for all $M \succ N$ and all $(p, t, x) \in \mathbf{R}^n \times \mathbf{R} \times \Omega$. The operator $F[\cdot]$ is uniformly elliptic if there is a constant $c_1 > 0$ such that

$$F(M, p, t, x) = F(N, p, t, x) \geq c_1 \operatorname{trace}(M - N) \tag{2.3}$$

for all $M \succ N$ and $(p, t, x) \in \mathbf{R}^n \times \mathbf{R} \times \Omega$

Definition 2.2. *The operator $F[\cdot]$ is nonincreasing if*

$$F(M, p, t, x) \leq F(M, p, s, x) \tag{2.4}$$

for all $t \geq s$ and $(M, p, x) \in S(n) \times \mathbf{R}^n \times \Omega$. The operator $F[\cdot]$ is decreasing if there is a constant $c_2 > 0$ such that

$$F(M, p, t, x) - F(M, p, s, x) \leq c_2(s - t) \tag{2.5}$$

for all $t > s$ and $(M, p, x) \in S(n) \times \mathbf{R}^n \times \Omega$.

Definition 2.3. *The operator $F[\cdot]$ is Lipschitz in p if there is a constant $c_3 > 0$ such that*

$$F(M, p, t, x) - F(M, q, t, x) \leq c_3|p - q| \tag{2.6}$$

for all $\quad (M, p, q, t, x) \in S(n) \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \times \Omega$

The operator $F[\cdot]$ is uniformly continuous in x if there is a continuous increasing function $\sigma(x)$ such that $\sigma(0) = 0$ and

$$F(M, p, t, x) - (F(M, p, t, y) \leq \sigma(|x - y|) \tag{2.7}$$

for all $\quad (M, p, t, x, y) \in S(n) \times \mathbf{R}^n \times \mathbf{R} \times \Omega \times \Omega$

Definition 2.4. *$w \in C(\Omega)$ is a viscosity supersolution of (1.1) if*

$$F(M, p, w(x), x) \leq 0 \quad \text{for all } (p, M) \in D^- w(x) \text{ and all } x \in \Omega \tag{2.8}$$

$w \in C(\Omega)$ is a viscosity subsolution of (1.1) if

$$F(M, p, w(x), x) \geq 0 \quad \text{for all } (p, M) \in D^+ w(x) \text{ and all } x \in \Omega \tag{2.9}$$

$w \in C(\Omega)$ is a viscosity solution of (1.1) if both (2.8) and (2.9) hold, where $D^+ w(x)$ and $D^- w(x)$ denote superdifferential and subdifferential of $w(x)$, respectively (see [1]).

Lemma 2.5. *Let $w \in C(\Omega)$. The following are equivalent:*

(i) w is a viscosity supersolution of (1.1);

(ii) $F(D^2\varphi(x_0), D\varphi(x_0), \varphi(x_0), x_0) \leq 0$ for all open set $G \subset \Omega$ and all $(x_0, \varphi) \in G \times C^\infty(G)$ such that $w(x) \geq \varphi(x)$ for all $x \in G$, $w(x_0) = \varphi(x_0)$.

The proof of Lemma 2.5 is similar to that of Lemma 2.15 in [1].

Lemma 2.6. *Let $w \in C(\Omega)$. The following are equivalent:*

(i) w is a viscosity subsolution of (1.1);

(ii) $F(D^2\varphi(x_0), D\varphi(x_0), \varphi(x_0), x_0) \geq 0$ for all open set $G \subset \Omega$ and all $(x_0, \varphi) \in G \times C^\infty(G)$ such that $w(x) \leq \varphi(x)$ for all $x \in G$, $w(x_0) = \varphi(x_0)$.

Definition 2.7. *For all $\varepsilon \in (0, \bar{\varepsilon}_0]$ ($\bar{\varepsilon}_0$ is the range in the implicit function Theorem, see [1]), we define*

$$F_\varepsilon^\pm(M, p, t, x) = F(M, p, t, x \pm \varepsilon p/(1 + |p|^2)^{\frac{1}{2}}) \tag{2.10}$$

for all $(M, p, t, x) \in S(n) \times \mathbf{R}^n \times \mathbf{R} \times \Omega$. It is very clear that $F_\varepsilon^\pm \in C(S(n) \times \mathbf{R}^n \times \mathbf{R}$