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A PRIORI ESTIMATES FOR POSITIVE SOLUTIONS OF SEMI-LINEAR ELLIPTIC SYSTEMS

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In this paper we obtain a priori estimates for positive solutions of certain elliptic systems by the blow-up method, and then by the degree theory we prove the existence of positive solution of these systems.

We study the elliptic system:

$$\begin{cases} -\Delta u^i = f^i(x, u), \text{ in } \quad \Omega, \ 1 \leq i \leq N \\ u|_{\partial\Omega} = 0, \quad u^i > 0, \text{ in } \quad \Omega, \ 1 \leq i \leq N \end{cases} \tag{1}$$

where Ω is an open bounded subset of R^n, $n \geq 3$, with smooth boundary $\partial\Omega$, and

$$u(x) \in R^N, \qquad f(x, t) \in [C^1(\overline{\Omega} \times R^N)]^N$$

Under the following three assumptions on $f(x, t)$, we prove the existence of a nontrivial positive solution of system (1):

(I) $\exists\ \alpha,\ 1 < \alpha < \frac{n+2}{n-2}$, and $h(x) \in [C(\overline{\Omega})]^N$ such that $\lim\limits_{|u| \to +\infty} \frac{f^i(x, u)}{|u|^\alpha} =$ $h^i(x) > 0$, on $\overline{\Omega}$.

(II) $\exists\ \delta > 0,\ \beta = \lambda_1/\sqrt{N}$, where λ_1 is the first eigenvalue of the operator $(-\Delta)$, such that $f^i(x, u) < \beta|u|$, for $|u| < \delta$.

(III) $f^i(x, t)$ is strictly positive for $|t| \neq 0$.

The method we used is based on that due to Spruck and Gidas (of. [2], [4]), in which a semi-linear equation was studied. The main point is to set up Liouville theorems of positive solutions both on the half space with Dirichlet boundary condition and on the whole space. A special system is introduced to our corresponding equation such that the blow-up method and the Maximum Principle can be applied.

This paper is devided into three sections. In the first section we study the symmetry of positive solutions for elliptic systems. Lioville theorems are proved in the second section. The third section is devoted to obtain a priori estimates and the existence of a nontrivial positive solution of system (1).

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1. Symmetry of positive solutions for elliptic systems

We start with discussing the symmetry of positive solutions for elliptic systems. Namely,

Theorem 1. 1. *Let $u \in C^2(\bar{B})$ be a positive solution of system (1), suppose that $f(x, u)$ satisfies*

a) $f(x, 0) = 0$,

b) $f^i(x, u) = f^i(|x|, u)$, *and each of $f^i(|x|, u)$ is monotonely decreasing with respect to $|x|$,*

c) $f^i_{u^j}(x, u) \geq 0$, *for $i \neq j$.*

Then $u(x) = u(|x|)$.

The following lemmas are simple applications of the Maximum Principle for equations.

Lemma 1. 2. *Let $u(x)$ be given as in theorem1. 1. Given a unit vector $v \in R^n$, suppose that $x_0 \in \partial B$, with $(x_0, v) > 0$, then $\exists\ \delta > 0$ such that*

$$\frac{\partial u^i}{\partial v}(x) < 0, \ for \quad any \quad x \in B_\delta(x_0) \cap \bar{B} \tag{2}$$

Proof. For any $i = 1, 2, \ldots, N$, according to assumptions a) , c) , in theorem 1. 1, we get

$$-\Delta u^i = f^i(x, u) - f^i(x, 0)$$

$$= \int_0^1 f^i_{u^j}(x, tu)\, u^j dt \geq \int_0^1 f^i_{u^i}(x, tu)\, dt \cdot u^i$$

Then lemma 1. 2 follows from the Hopf Maximum Principle for elliptic equations.

Lemma 1. 3. *Let v be given as above. Set*

$$\sum_\lambda = \{x \in B, \ (x, v) > \lambda\}$$

$$\sum_\lambda' = \{x^\lambda = x - 2[(x, v) - \lambda]v, \ x \in \sum_\lambda\}$$

$$T_\lambda = \{x \in B, \ (x, v) = \lambda\}$$

where $\lambda > 0$. If $u(x)$ is a positive solution of system (1) which satisfies assumptions in theorem 1. 1, and

$$\begin{cases} u^i(x) \leq u^i(x^\lambda) & x \in \sum_\lambda \\ \dfrac{\partial u^i(x)}{\partial v} \leq 0 & x \in T_\lambda \end{cases}$$

Then