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A Projection Preconditioner for Solving the Implicit Immersed Boundary Equations

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Abstract. This paper presents a method for solving the linear semi-implicit immersed boundary equations which avoids the severe time step restriction presented by explicit-time methods. The Lagrangian variables are eliminated via a Schur complement to form a purely Eulerian saddle point system, which is preconditioned by a projection operator and then solved by a Krylov subspace method. From the viewpoint of projection methods, we derive an ideal preconditioner for the saddle point problem and compare the efficiency of a number of simpler preconditioners that approximate this perfect one. For low Reynolds number and high stiffness, one particular projection preconditioner yields an efficiency improvement of the explicit IB method by a factor around thirty. Substantial speed-ups over explicit-time method are achieved for Reynolds number below 100. This speedup increases as the Eulerian grid size and/or the Reynolds number are further reduced.

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Key words: Fluid-structure interaction, immersed boundary method, projection method, preconditioning.

1. Introduction

The Immersed Boundary (IB) method introduced by Peskin [19, 20] has been a popular approach for simulating fluid-structure interactions. Physical variables for the fluid are discretized on an Eulerian grid while those for the immersed boundary are discretized on a Lagrangian grid. The fluid satisfies the no-slip condition on the immersed

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boundary, which means that the Lagrangian grid points move at a velocity interpolated from the Eulerian grid. Deformations of the immersed boundary generate elastic forces which are transmitted to the fluid through a forcing term added to the governing equations of fluid dynamics. In this manner the IB method provides much flexibility in modeling the coupling between the Eulerian and Lagrangian variables, since explicitly enforcing boundary conditions at the fluid-structure interface is avoided.

The popularity of the IB method is partly due to its simplicity. In a typical explicit-time method, the Eulerian velocity and pressure fields are updated for a fixed configuration of the immersed structure, and then the position of the Lagrangian structure is updated from the newly computed velocity field. This approach effectively decouples the Eulerian and Lagrangian equations, and solvers are needed only for the Eulerian equations (i.e., the incompressible Stokes or Navier-Stokes equations), for which fast Cartesian grid solution methods are available. The implementation is straightforward since it only entails augmenting one's favorite fluid solver with the IB forcing term. Nonetheless, when an elastic boundary becomes stiff, explicit-time IB methods suffer from either instability or restrictively small time steps.

To remedy the severe time step restriction of explicit IB methods, a number of implicit and semi-implicit schemes have been developed. However, their implementation is much more involved and is a subject of ongoing research; see for example [3, 4, 11, 13, 15–18, 22, 24] and references therein. These methods are centered at answering two essential questions:

(A) How does stiffness affect the stability of the numerical solver?

(B) How to efficiently solve the discretized equations that are highly stiff?

Clearly these two questions are closely related. It had been commonly believed that only fully implicit discretizations could produce an unconditionally stable IB method until the work of Newren *et. al.* [17]. They showed that semi-implicit versions of backward Euler and Crank-Nicolson schemes can be made stable so long as the spreading and interpolation operators are evaluated at the same time instant and the same spatial location. When this is satisfied, the total energy of the numerical system does not increase over time even if the evaluation of the spreading and interpolation operators are lagged in time. This conclusion not only answers question (A), but also partially answers question (B), since the lagged evaluation of the spreading and interpolation operators opens up exciting possibilities of unconditionally stable discretizations via *linear* systems.

Many implicit methods use a Schur complement approach to reduce the coupled Eulerian-Lagrangian equations to purely Lagrangian equations [3, 4, 16]. These methods achieve a substantial speed-up over explicit methods when there are relatively few Lagrangian mesh nodes. In addition, some methods require that the boundaries be smooth, closed curves [13]. An open question is whether there exist robust, general-purpose implicit methods that are more efficient than explicit methods, or whether specialized methods must be developed for specific problems.