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AN IMMERSED EULERIAN-LAGRANGIAN LOCALIZED ADJOINT METHOD FOR TRANSIENT ADVECTION-DIFFUSION EQUATIONS WITH INTERFACES

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Abstract. We develop and analyze an immersed Eulerian-Lagrangian localized adjoint method (ImELLAM) for transient advection-diffusion equations with interfaces. The derived method possesses the combined advantages of the immersed finite element method and the Eulerian-Lagrangian method.

Key Words. advection-diffusion problem, Eulerian-Lagrangian method, error estimate, immersed finite element method, interfaces.

1. Introduction

Transient advection-diffusion equations arise in mathematical models for describing petroleum reservoir simulation, groundwater contaminant transport, geological storage of carbon dioxide and remediation, and many other applications [1, 13, 2, 7, 12, 13, 14, 20]. These equations admit solutions with moving steep fronts and complicated structures. Furthermore, subsurface porous medium matrix often contains a variety of faults and fractures of different magnitude. Those relatively large faults must be accurately incorporated into the corresponding mathematical models, in which the geological formations consist of several subdomains with different geological properties and salient physical interfaces. This also means that in the numerical discretization the computational meshes must align with the large faults in order to obtain a stable and accurate numerical solution. Note that the number of large faults is usually quite limited, so the modeling and numerical implementation is doable. On the other hand, there are numerous relatively small fractures which are very difficult, if not impossible at all, to describe in a deterministic manner geologically. As a matter of fact, these relatively tiny fractures are often described in a probability sense. The impact of these tiny fractures can be handled via the approach of upscaling or multiscale numerical techniques. As for those intermediate fractures, they are probably too big to be upscaled into the underlying numerical schemes in any reasonable manner. On the other hand, there are probably too many intermediate fractures such that the computational meshes of the underlying numerical scheme align with each of them. Based on these considerations we plan to adopt the approach of immersed numerical method to handle these intermediate fractures.

To expose the idea, in this paper we consider the one-dimensional transient linear advection-diffusion equation with interfaces

$$\begin{aligned} \phi u_t + (Vu - Du_x)_x &= f(x,t), \qquad x \in (a,b), \quad 0 < t \le T, \\ u(x,0) &= u_0(x), \qquad x \in [a,b]. \end{aligned} \tag{1}$$

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In such areas as porous medium flow and transport, the geological formations may consist of several subdomains with different geological properties. Consequently, there exist physical interfaces between different subdomains. Across these interfaces, the concentration $u(x,t)$ and the Darcy flux $V(x)$ are continuous, but the porosity of the porous medium $\phi(x)$ and the diffusion coefficient $D(x)$ are discontinuous. Nevertheless, the diffusive flux is continuous across these interfaces. We assume that V is constant in the domain (a,b), ϕ and D are piecewise constants and $a < \alpha_1 < \cdots < \alpha_K < b$ are the interfaces. This leads to the following interface conditions for $k = 1, \cdots, K$,

$$[\![u]\!](\alpha_k, t) = 0, \qquad [\![Du_x]\!](\alpha_k, t) = 0, \qquad t \in [0, T], \tag{2}$$

where $[\![u]\!](\alpha_k, t) = u(\alpha_k^+, t) - u(\alpha_k^-, t)$ represents the jump of u across the interface $x = \alpha_k$. To focus on main idea for the development and the analysis of the ImELLAM scheme, we assume that the problem is closed by the periodic boundary condition.

In this paper we develop and analyze an immersed Eulerian-Lagrangian localized adjoint method (ImELLAM) for transient advection-diffusion equations with interfaces. The rest of the paper is organized as follows: In §2 we present some preliminaries that are needed in the development and analysis of the ImELLAM scheme. In §3 we derive the ImELLAM scheme. In §4 we prove an optimal-order error estimate for the ImELLAM scheme. §5 contains concluding remarks.

2. Preliminary

In this section we recall some preliminaries that are needed in the development and analysis of the ImELLAM scheme.

2.1. Sobolev Spaces. Let $W_p^k(a,b)$ consist of functions whose weak derivatives up to order-k are p-th Lebesgue integrable in (a,b), and $H^k(a,b) := W_2^k(a,b)$. Let $H_E^m(a,b)$ be the subspace of $H^m(a,b)$ with periodic boundary condition. For any Banach space X, we introduce Sobolev spaces involving time [6]

$$\begin{aligned}
W_p^k(t_1,t_2;X) &:= \left\{ f : \left\| \frac{\partial^l f}{\partial t^l}(\cdot,t) \right\|_X \in L^p(t_1,t_2),\ 0 \le l \le k,\ 1 \le p \le \infty \right\}, \\
\|f\|_{W_p^k(t_1,t_2;X)} &:= \begin{cases} \left(\displaystyle\sum_{l=0}^{k} \int_{t_1}^{t_2} \left\| \frac{\partial^l f}{\partial t^l}(\cdot,t) \right\|_X^p dt \right)^{1/p}, & 1 \le p < \infty, \\ \displaystyle\max_{0 \le l \le k} \operatorname*{esssup}_{t \in (t_1,t_2)} \left\| \frac{\partial^l f}{\partial t^l}(\cdot,t) \right\|_X, & p = \infty. \end{cases}
\end{aligned}$$

We also introduce piecewise-smooth Sobolev spaces incorporated with certain continuity conditions and the corresponding norms for the immersed finite element method [10, 15]

$$\begin{aligned}
PW_p^k(a,b) &:= \Big\{ v : \ v|_{(\alpha_{k-1},\alpha_k)} \in W_p^k(\alpha_{k-1},\alpha_k), k = 1, \cdots, K+1 \Big\}, \\
PH_{int}^2(a,b) &:= \Big\{ v : \ v \in C(a,b), \quad v|_{(\alpha_{k-1},\alpha_k)} \in H^2(\alpha_{k-1},\alpha_k), \\
&\qquad\quad [\![Dv_x]\!](\alpha_k) = 0, \quad k = 1, \cdots, K+1 \Big\}.
\end{aligned}$$