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A POSTERIORI ERROR ESTIMATES OF *hp*-FEM FOR OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we investigate a posteriori error estimates of the *hp*-finite element method for a distributed convex optimal control problem governed by the elliptic partial differential equations. A family of weighted a posteriori error estimators of residual type are formulated. Both reliability and efficiency of the estimators are analyzed.

Key Words. *hp*-finite element method, optimal control problem, a posteriori error estimates

1. Introduction

Finite element approximation plays an important role in the numerical methods of optimal control problems. There have been extensive theoretical and numerical studies for the finite element approximation of various optimal control problems. However the literature is too huge to even give a very brief review here. In recent years, the adaptive finite element method has been extensively investigated. Adaptive finite element approximation is among the most important means to boost the accuracy and efficiency of the finite element discretizations. It ensures a higher density of nodes in certain areas of the given domain, where the solution is more difficult to approximate, using an a posteriori error indicator. We acknowledge the pioneering work due to Babuška and Rheinboldt [4]. Further references can be found in the monographs [2], [5], [31], and the references cited therein.

In the recent years, adaptive finite elements for optimal control has become a focus of research interests. There have appeared many research papers on the adaptivity of various optimal control problems. For example, [6] studied the adaptive finite element method for optimal control problems via a goal-orientated approach, while a posteriori error estimates of residual type were derived for convex distributed optimal control problems governed by the elliptic and the parabolic equations in [18], [22]-[24], and for boundary control problems in [21].

To authors' knowledge, the papers discussing the adaptive finite element methods for optimal control problems are all related to low order FEM, i.e. h-FEM. In the adaptive h-FEM, the adaptivity is performed by mesh refinement guided by a posteriori error estimators. There are also many high order methods, such as spectral element methods, the p-version and the hp-version finite element methods, which have been applied to many practical problems. Using the local refinement of the meshes where the solution is singular and applying higher order polynomials where the solution is smooth, the adaptive hp-version finite element method can

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achieve very high computation efficiency. There have been some extensive investigations of adaptive hp-FEM for the elliptic partial differential equations (see, e.g., [3], [7], [10], [15], [26], [28] and [29]).

It seems to be very suitable to apply the hp finite element method to approximate optimal control problem, see [20]. The main objective of this paper is to establish a posteriori error estimates for the hp-version finite element approximation of a model optimal control problem governed by the elliptic partial differential equation, which were not available before and can be used to guide the hp-adaptivity process. In this paper, we proved the upper and lower bounds of the a posteriori error estimates, although there is a gap of order p^2 between the lower and the upper bounds due to the existing gap in the a posteriori estimates for the hp-adaptive finite element approximation of the elliptic equations. We also formulate a family of a posteriori error estimators given by weighted residuals on the elements and the edges. In our work, we used some techniques that have been used for a posteriori error estimates of the h-version FEM for optimal control problems (see, e.g., [22]-[24] for more details). We also used the weighted techniques and some estimates of the hp-interpolation of Clément-type proposed in [26], where a posteriori error estimates were obtained for hp-FEM of the elliptic partial differential equations. In comparison with the hp a posteriori error estimates for the elliptic equations, the main difference here is how to handle the variational inequality in the optimality conditions as in the h-version adaptive finite element method for optimal control. Besides different interpolators and interpolation results that now have to be used, the variational inequality is further different from that for the control constraint of obstacle type in the literature, due to the different control constraint set studied in this paper. While the existing techniques for the constraints of obstacle type can be modified for deriving the upper bounds, the techniques are very different to derive the lower bound here. These are studied in Lemma 5.2 where we use the inverse inequality to estimate the lower bound for the control, and this approach has not been used before.

The paper is organized as follows: In Section 2, we introduce the model problem and its weak formulation, and give the hp-finite element spaces and the hp-finite element approximation of the control problem. In Section 3, some technical lemmas are introduced, which are used for the later a posteriori error analysis. In Section 4, an a posteriori error estimator for the control problems is provided based on the local residual technique. It is shown that the a posteriori error estimator is an upper bound of the error. In Section 5, it is proved that the a posteriori error estimator provided in Section 4 is also a lower bound of the error, although there is a gap of order p^2 between the lower and the upper bounds. In the last section, a family of weighted a posteriori error estimators are presented as the extension of the analysis of Sections 4 and 5 using the weight function technique introduced in [26].

2. The model problem and hp-FEM approximation

Let $\Omega(\Omega_U)$ be a bounded domain in R^2 with the Lipschitz boundary $\partial\Omega(\partial\Omega_U)$. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $||\cdot||_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)(W_0^{m,2}(\Omega))$ by $H^m(\Omega)(H_0^m(\Omega))$. In addition, c or C denotes a general positive constant independent of h and p.