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NUMERICAL METHODS FOR NON-SMOOTH L^1 OPTIMIZATION : APPLICATIONS TO FREE SURFACE FLOWS AND IMAGE DENOISING

ALEXANDRE CABOUSSAT, ROLAND GLOWINSKI, AND VICTORIA PONS

Abstract. Non-smooth optimization problems based on L^1 norms are investigated for smoothing of signals with noise or functions with sharp gradients. The use of L^1 norms allows to reduce the blurring introduced by methods based on L^2 norms. Numerical methods based on over-relaxation and augmented Lagrangian algorithms are proposed. Applications to free surface flows and image denoising are presented.

Key Words. L^1 optimization, Over-relaxation algorithm, Augmented Lagrangian methods, Smoothing, Image Denoising.

1. Introduction

The need to smooth a given function is a problem that arises in many fields of science and engineering. A trade-off between the conservation of the accuracy and the regularity properties must be obtained. In volume-of-fluid methods pertaining to computational fluid dynamics, the smoothing of volume fractions of materials is required when calculating interfacial effects [2, 16]. In image treatment, noise can be removed by the application of appropriate filters, based on average mean calculations, low/high-pass filters or PDE-based techniques. Classical smoothing techniques range from kernel-based methods [2], to PDE-based techniques or wavelet-based methods [9]. However when using classical techniques, based on quadratic or L^2 norms, blurring of the sharp edges is often introduced. Recently, methods based on L^1 distances have received a lot more attention in various settings [4, 8, 9, 12, 19, 20]. More generally, smoothing is required when a numerical approximation of the derivatives of a non-smooth function is needed.

In this article, numerical methods for non-smooth optimization problems relying on L^1 norms are presented in order to reduce the blurring due to quadratic terms in classical methods. The solution methods for the smoothing of a given signal require advanced techniques since strict convexity and differentiability properties are not satisfied. Moreover, the uniqueness of the solution is not guaranteed, unless some regularization terms are introduced [15, 21].

The problems addressed here consist of the minimization of the distance between a given signal, typically with jumps or noise, and a smooth approximation whose first derivatives are regular. The L^1 distance is considered first. A smoothing term is introduced to add regularity. The regularization term is given either by the L^2 norm or the L^1 norm of the gradient of the approximated solution. Finally the L^2 distance is considered together with a L^1 smoothing term with bounded variation. Efficient numerical techniques are proposed for the solution of each of

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these problems. The space discretization is addressed with piecewise linear finite elements. The discretized optimization problems are solved with either an over-relaxation algorithm [17], or an augmented Lagrangian approach [17, 18] when the strict convexity property is not satisfied, or a combination of both.

Numerical results are presented for two kinds of applications. First the smoothing of volume fractions in volume-of-fluid algorithms for multiphase flows is known to introduce artificial numerical errors near the boundaries of the physical domain (spurious currents) [2, 3, 16, 24, 27, 28]. The approximation of the surface tension effects near the boundaries requires for instance the introduction of *ghost cells* outside the domain [13]. This drawback can be corrected by the proposed approach.

On the other hand, image denoising and reconstruction is a very active field of research [6, 8, 10, 25]. The use of L^1 distance has two main properties: it allows to avoid the blurring of edges due to quadratic regularization terms, while being appropriate for removing the noise. Numerical examples based on a famous example (see *e.g.* [10]), are presented to compare the suggested approaches.

2. Non-Smooth Optimization Models

Let Ω be a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$. Let $f \in L^2(\Omega)$ be a given function (or *signal*), that contains either sharp interfaces, discontinuities along lines or points, or noise. We want to approximate the signal f by a smooth function u (typically $u \in H^1(\Omega)$) in order to (i) be able to approximate the derivatives of f through the derivatives of the function u, or (ii) remove the noise from the original signal.

Let $\Omega \subset \mathbb{R}^2$ be bounded with partition of the boundary $\Gamma_0 \cup \Gamma_1 = \partial\Omega$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let us denote by V_0 and W_0 the spaces

$$\begin{aligned} V_0 &= \left\{ v \in H^1(\Omega) \, : \, v|_{\Gamma_0} = 0 \right\}, \\ W_0 &= \left\{ v \in W^{1,1}(\Omega) \, : \, v|_{\Gamma_0} = 0 \right\}. \end{aligned}$$

The Neumann case $\Gamma_0 = \emptyset$ and $\Gamma_1 = \partial\Omega$ is also included. We consider three possible approaches: first the L^1 distance between the original function and its smooth approximation is considered, together with a regularization term depending on the gradient of the approximation. This regularization term can be taken as the L^2 or the L^1 norm of the gradient. The use of the L^1 distance allows to conserve the sharp gradient (edges) of the original function. Finally, we consider the L^2 distance, together with a L^1 smoothing term, and design adequate numerical methods for each of these problems.

2.1. Optimization with L^1 Distance and L^2 Smoothing Term.

For $f \in L^2(\Omega)$, solve

$$\min_{v \in V_0} \int_\Omega |v - f| \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla v|^2 \, dx. \tag{1}$$

The distance term $\int_\Omega |v - f| \, dx$ is not differentiable, but the addition of the smoothing term $\frac{\varepsilon}{2} \int_\Omega |\nabla v|^2 \, dx$ forces uniqueness through (strict) convexity. The following theorem holds:

Theorem 1. *Problem (1) admits a unique solution $u \in V_0$ (also if $\Gamma_0 = \emptyset$). The solution is characterized by*

$$\varepsilon \int_\Omega \nabla u \cdot \nabla (v - u) dx + \int_\Omega |v - f| \, dx - \int_\Omega |u - f| \, dx \geq 0, \quad \forall v \in V_0. \tag{2}$$