INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 3, Pages 402–419

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Computing and Information

NUMERICAL SOLUTION OF A NON-SMOOTH VARIATIONAL PROBLEM ARISING IN STRESS ANALYSIS : THE SCALAR CASE

ALEXANDRE CABOUSSAT AND ROLAND GLOWINSKI

Abstract. A non-smooth constrained minimization problem arising in the stress analysis of a plastic body is considered. A numerical method for the computation of the load capacity ratio is presented to determine if the elastic body fractures under external traction. In the scalar case, the maximum principle allows one to reduce the problem to a convex one under linear constraints. An augmented Lagrangian method, together with an approximation by finite elements is advocated for the computation of the load capacity ratio and the corresponding elastic stress. The generalized eigenvalues and eigenvectors of the corresponding operator are computed for various two-dimensional bodies and fractures are discussed.

Key Words. Non-smooth optimization, Stresses analysis, Augmented Lagrangian method, Finite elements approximation, Elasticity theory.

1. Introduction and Motivations

The numerical solution of a non-smooth minimization problem arising in the stress analysis of a plastic body is investigated. The problem of interest is the computation of the maximal stress of an elastic three-dimensional body under external forces [3, 18]. The computation of that threshold allows one to determine if the material can support the forces applied to it or if the traction on the material or its boundary is too large.

Consider a homogeneous isotropic elastic body $\Lambda \subset \mathbb{R}^3$. The *load capacity ratio*, as defined in [16], is the maximal positive number C such that the body will not collapse under any external traction field bounded by CY_0, where Y_0 is the elastic limit. It is independent of the distribution of external loading. If we neglect the body forces and consider only forces on the boundary, it implies that no collapse will occur for any field $\mathbf{g}$ on $\partial\Lambda$ as long as ess $\sup_{y\in\partial\Lambda} |\mathbf{g}(y)| < CY_0$. The load capacity ratio is a quantity that depends on the geometry of the domain Λ.

The load capacity ratio is a quantity that also appears in the *limit analysis problem* in Temam (1985) [18], and which represents, from the mechanical standpoint, a criterion to determine if the material can support the imposition of both body and surface external forces.

Load capacity ratio and *stress concentration factors* are used by engineers to compare the maximal stress for a given body with the stress computed analytically for simplified geometries [13, 15]. Such problems also lead to the computation of optimal stresses as in [17, 18].

Received by the editors October 30, 2008 and, in revised form, January 5, 2009.
2000 *Mathematics Subject Classification.* 65K10, 65N30, 49S05, 74G70 .

In this paper, we consider the situation without body forces (see [14, 18] for a theoretical investigation of the addition of body forces). Let us assume that the boundary of Λ is sufficiently smooth and can be decomposed into $\gamma_0 \cup \gamma_1 \cup \gamma_2$, such that $\gamma_i \cap \gamma_j = \emptyset$, $i, j = 0, 1, 2$, $i \neq j$. An external surface traction acts on the boundary γ_1, while the body is fixed on the boundary γ_0.

When the domain Λ is an infinite cylinder oriented along one direction of space and the external surface traction is oriented along that direction, the limit analysis problem can be written as a scalar problem. The numerical approximation of the scalar case is considered here, namely to compute the load capacity ratio when considering a surface traction field oriented along the invariant direction of Λ. Following [1, 2], a numerical method for the approximation of the inverse $\delta := C^{-1}$ of the load capacity ratio is proposed.

Let Ω be the two-dimensional domain obtained by cutting Λ perpendicularly to its invariance direction, and $\Gamma_i = \gamma_i \cap \bar{\Omega}$, $i = 0, 1, 2$. Our aim is therefore to compute the quantity

$$\delta = \inf_{v \in \Sigma} \int_\Omega |\nabla v| \, dx,$$

where $\Sigma = \left\{ v \in V_0, \ \displaystyle\int_{\Gamma_1} |v| \, dS = 1 \right\}$ and $V_0 = \left\{ v \in H^1(\Omega), \ v = 0, \ \text{on } \Gamma_0 \right\}$.

Such non-smooth optimization problems require appropriate solution methods [1, 4, 9, 10]; they are related to the numerical approximation of the degenerated eigenvalues of non-smooth operators [2, 12]. In this article, an augmented Lagrangian method is advocated and applied to two-dimensional domains, without introducing any regularization or convexification parameters [8, 11, 12] (other than the regularization coming from the space approximation).

In Section 2, the model problem is derived and the scalar problem is justified; the maximum principle is used to transform the problem into a convex optimization problem under linear constraints. Section 3 presents an augmented Lagrangian algorithm for the solution of such non-smooth optimization problems that takes advantage of linearity properties. The finite element discretization is detailed in Section 4. Numerical results in various settings are finally given in Section 5, together with the numerical investigation of fractures when modifying the boundaries under traction and the fixed boundaries.

2. Modeling of Elastic Bodies and the Limit Analysis Problem

Let us consider a elastic material body that occupies a domain $\Lambda \subset \mathbb{R}^3$. The smooth boundary of the domain Λ is partitioned into $\partial\Lambda = \gamma_0 \cup \gamma_1 \cup \gamma_2$ such that $\gamma_i \cap \gamma_j = \emptyset$, $i \neq j$. The elastic material is under body forces $\mathbf{f} \in L^2(\Lambda)^3$ and boundary forces $\mathbf{g} \in L^2(\gamma_1)^3$. The variational problem for the computation of the elastic displacement of the body reads as follows

$$\inf_{\mathbf{v} \in \mathbf{V}_0} \left[\int_\Lambda \psi(\mathbf{D}(\mathbf{v})) dV - L(\mathbf{v}) \right], \tag{1}$$

where $\mathbf{D}(\mathbf{v}) = \dfrac{1}{2} \left(\nabla\mathbf{v} + \nabla\mathbf{v}^T \right)$ is the deformation tensor, $\psi(\cdot)$ is a proper, lower semi-continuous convex function that characterizes the material properties,

$$L(\mathbf{v}) \ = \ \int_\Lambda \mathbf{f} \cdot \mathbf{v} dV + \int_{\gamma_1} \mathbf{g} \cdot \mathbf{v} dS,$$