INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 3, Pages 420–439

DIMENSION SPLITTING METHOD FOR 3D ROTATING COMPRESSIBLE NAVIER-STOKES EQUATIONS IN THE TURBOMACHINERY

KAITAI LI AND DEMIN LIU

Dedicated to Professor Roland Glowinski on the occasion of his 70th birthday

Abstract. In this paper, we propose a dimension splitting method for Navier-Stokes equations(NSEs). The main idea is as follows. The domain of flow in 3D is decomposed into several thin layers. In each layer, The 3D NSEs can be represented as the sum of a membrane operator and a normal (bending) operator on the boundary of layer. And The Euler central difference is used to approximate the bending operator. When restricting the 3D NSEs on the boundary in each layer, we obtain a series of two dimensional-three components NSEs (called as 2D-3C NSEs). Then we construct an approximate solution of 3D NSES by solutions of those 2D-3C NSEs.

Key Words. 2D Manifold, Semi-Geodesic Coordinate Navier-Stokes Equations, Dimension Splitting Method.

1. Introduction

In [1, 2], the authors studied two dimensional flow on the stream surface, derived a nonlinear boundary value problem satisfied by stream function defined on the stream surface, and studied its finite element approximation. In [3, 4], Kaitai Li propose a dimensional splitting method for the linearly elastic shell based on differential geometry and tensor analysis. In this paper we will use classical tensor calculation to propose a new method , called "dimensional splitting method" for 3D rotating NSEs (compressible or incompressible).

The main idea is that, a 3D flow domain Ω bounded by four 2D-surfaces is decomposed into several thin layers Ω^i_{i-1} bounded by 2D surfaces $\Im_i,\ i=1,\ 2,\ \cdots,\ m$. 3D rotating Navier-Stokes operators in thin layer $\Omega^i_{i-1}\cup\Omega^{i+1}_i$ under local semi-geodesic coordinate based on the surface $\Im_i$ can be represented into the sum of a membrane operator on $\Im_i$ and a normal (bending) operator to $\Im$, then applying Euler central difference approximate bending operator. Then we obtain a restriction of 3D rotating NSEs on the $\Im_i$, that is a three components-two dimensional NSEs (called 2D-3C NSEs). Solving 2D-3C NSEs on $\Im_i,\ i=1,\ \cdots,\ m$ by parallel algorithms and reiterating until convergence , we can obtain approximate solution of 3D rotating NSEs. It is obvious that the method is different from the classical domain decomposition method because we only solve a two dimensional problem in each sub-domain(stream surface layer), instead of solving a 3D problem, and the 3D domain is decomposed into sub-domains by two dimensional manifold instead

Received by the editors October 30, 2008 and, in revised form, January 5, 2009.
2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.
This research was supported by NSFC project 10571142, 10771167, 10871156.

of flat plane. In addition , this paper provide three methods to solve 2D-3C NSE, those are artificial viscous method, streamline-FEM and stream functions methods.

The contents are organized as following : provide the mathematical description of the blade's surface in section 1; a domain partition's method and rotating NSEs in semi-geodesic coordinate based on two dimensional manifold $\Im$ in section 2; a 2D-3C NSEs-a restriction of 3D rotating Navier-Stokes equation to $\Im$ in section 4; provide a Korn's inequality on the $\Im$ in section 5; prove the existence of solution to corresponding variational formulation in section 6.

2. Geometry of the Channel in the Impeller and Navier-Stokes Equations

Let us consider the geometry of the channel Ω_ε bounded by two blade's surfaces $\Gamma_s^+,\ \Gamma_s^-$ and top- and bottom- surfaces $\Gamma_t,\ \Gamma_b$ in a impeller. Let $D\subset\Re^2$ simply-connected open subset of $\Re^2$, $\mathbf{E}$ denotes a three-dimensional Euclidean space. The surface of blade is a two dimensional manifold $\Im$ which is a smooth injective immersion $\vec{R}\in\ \mathbf{C}^3(D;\mathbf{E}^3)$:

$$D=\{(z,\ r)\}\subset R^2\Rightarrow R^3,\ \vec{R}(z,\ r)=r\vec{e}_r+r\Theta(z,\ r)\vec{e}_\theta+z\vec{k}, \tag{2. 1}$$

where $(\vec{e}_r,\ \vec{e}_\theta,\ \vec{k})$ are base vectors of cylindrical coordinate system rotating with the impeller and $(x^1=z,\ x^2=r)$ are the parameters describing the surface $\Im$ of blade as a submanifold embedding into $\mathbf{E}^3$, are also usually called Gaussian coordinate system on $\Im$.

In this case the Riemannian metric tensors of manifold $\Im$ are given by

$$\begin{cases} a_{\alpha\beta}=\frac{\partial\vec{R}}{\partial x^\alpha}\frac{\partial\vec{R}}{\partial x^\beta}=\frac{\partial r}{\partial x^\alpha}\frac{\partial r}{\partial x^\beta}+r^2\Theta_\alpha\Theta_\beta+\frac{\partial z}{\partial x^\alpha}\frac{\partial z}{\partial x^\beta}=\delta_{\alpha\beta}+r^2\Theta_\alpha\Theta_\beta,\\ a=\det(a_{\alpha\beta})=1+r^2(\Theta_1^2+\Theta_2^2), \end{cases} \tag{2. 2}$$

where

$$\Theta_\alpha=\frac{\partial\Theta}{\partial x^\alpha},$$

$b_{\alpha\beta}$ second fundamental form of the surface $\Im$

$$b_{\alpha\beta}=\frac{\partial^2\vec{R}}{\partial x^\alpha x^\beta}(\frac{\partial\vec{R}}{\partial x^1}\times\frac{\partial\vec{R}}{\partial x^2})/\sqrt{a}=\frac{1}{\sqrt{a}}\begin{vmatrix} x_{\alpha\beta} & y_{\alpha\beta} & z_{\alpha\beta}\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2 \end{vmatrix},$$

where $(x,\ y,\ z)$ denote Cartan coordinate, and $x_\alpha=\frac{\partial x}{\partial x^\alpha},\ y_\alpha=\frac{\partial y}{\partial x^\alpha},\ x_{\alpha\beta}=\frac{\partial^2 x}{\partial x^\alpha x^\beta},\ \cdots,$ Therefore

$$\begin{cases} b_{11}=\frac{1}{\sqrt{a}}(x^2\Theta_{11}+\Theta_2(a-1)),\\ b_{12}=\frac{1}{\sqrt{a}}(x^2\Theta_{12}+\Theta_1 a)=b_{21},\\ b_{22}=\frac{1}{\sqrt{a}}(x^2\Theta_{22}+\Theta_2(a+1)),\\ b=\det(b_{\alpha\beta})=b_{11}b_{22}-b_{12}^2. \end{cases} \tag{2. 3}$$

The mean curvature H and Gaussian curvature K are given by

$$2H=a^{\alpha\beta}b_{\alpha\beta}=\frac{1}{\sqrt{a}}\big(a_{11}b_{22}-2a_{12}b_{12}+a_{22}b_{22}\big),\qquad K=\frac{b}{a}. \tag{2. 4}$$

It is clear that

$$(a_{\alpha\beta})\in\mathbf{C}^2(D;\mathcal{S}^2_>),\quad(b_{\alpha\beta})\in\ \mathbf{C}^2(D;\mathcal{S}^2)$$

are two matrix fields where $\mathcal{S}^2$ and $\mathcal{S}^2_>$ denote the sets of all symmetric matrices of order two , and of all symmetric, positive definite matrices. $(a_{\alpha\beta}):D\to\mathcal{S}^2_>$ and $(b_{\alpha\beta}):D\to\mathcal{S}^2$ are the covariant components of the first and second fundamental forms of the surface $\Im$.