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The Totally Non-positive Matrix Completion Problem†

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Abstract. In this paper, the totally non-positive matrix is introduced. The totally non-positive completion asks which partial totally non-positive matrices have a completion to a totally non-positive matrix. This problem has, in general, a negative answer. Therefore, our question is for what kind of labeled graphs G each partial totally non-positive matrix whose associated graph is G has a totally non-positive completion? If G is not a monotonically labeled graph or monotonically labeled cycle, we give necessary and sufficient conditions that guarantee the existence of the desired completion.

Key words: Completion problem; totally non-positive matrix; patrial matrix; monotonically labeled graph.

AMS subject classifications: 15A48

1 Introduction

Matrix completion problems have attracted the attention of many researchers in recent years (see [1]-[9]). Such problems have a variety of applied motivations including seismic reconstruction problems entropy methods in statistical/physical, electrical systems/engineering, etc..

A partial matrix over R is an $n \times n$ array in which some entries are specified, while the remaining entries are free to be chosen from R. A completion of a partial matrix is the conventional matrix resulting from a particular choice of values for the unspecified entries, which will be denoted by x_{ij} or a question mark in this paper. A matrix completion problem asks which partial matrices have completions with some desired property.

In a class of matrix completion problem, such as the positive matrix completion (see [1]-[4]), P- matrix completion (see [5]-[6]), N- matrix completion (see [7]-[8]) and totally positive matrix completion (see [9]), etc., these matrices are defined in analogous way, that is, by their determinantal inequalities. So a new concept of the totally non-positive matrix is introduced here in the same thinking.

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Definition 1.1. An $n \times n$ real matrix is said to be totally non-positive if every minor of the matrix is non-positive.

Proposition 1.1. Let $A = (-a_{ij})$ be an $n \times n$ totally non-positive matrix.

1. If D is a positive diagonal matrix, then AD and DA are totally non-positive matrices.
2. If D is a positive diagonal matrix, then DAD^{-1} is a totally non-positive matrix.
3. Total non-positivity is not preserved by permutation similarity.
4. If the diagonal entry $a_{ii} < 0$ $(i = 1, 2, \cdots, n)$, then $a_{ij} < 0$,$i \neq j$.
5. Any submatrix of A is a totally non-positive matrix.

These properties allow us to assume that all specified entries of A are negative and all diagonal elements are equal to -1.

In spite of totally non-positivity not being preserved by permutation similarity, we can establish the following result.

Proposition 1.2. Let A be a totally non-positive matrix of size $n \times n$, and let P be the permutation matrix $P = [e_n, e_{n-1}, \cdots, e_2, e_1]$. Then PAP^T is a totally non-positive matrix.

The last property of Proposition 1.1 allows us to give the following definition.

Definition 1.2. A partial matrix is said to be a partial totally non-positive matrix if every completely specified submatrix is a totally non-positive matrix.

In this paper, we study the totally non-positive matrix completion problem, that is, when a partial totally non-positive matrix has a totally non-positive completion.

At first, we notice that not every partial totally non-positive matrix has a totally non-positive completion.

Example 1.1.

$$A = \begin{pmatrix} -1 & -x_{12} & -0.7 \\ -1 & -1 & -x_{23} \\ -x_{31} & -1 & -0.8 \end{pmatrix}. \tag{1}$$

The matrix A does not have a totally non-positive completion because if $detA[\{1,2\}|\{1,3\}] \leq 0$, then $x_{23} \leq 0.7$, and if $detA[\{2,3\}|\{2,3\}] \leq 0$, then $x_{23} \geq 0.8$, which is a contradiction.

Example 1.1 gives an non-combinatorially symmetric partial matrices. However, for combinatorially symmetric partial matrices, the problem has, in general, a negative answer.

Example 1.2.

$$B = \begin{pmatrix} -1 & -1 & -x_{13} & -2 \\ -2 & -1 & -1 & -x_{24} \\ -x_{31} & -2 & -1 & -3 \\ -1 & -x_{42} & -2 & -1 \end{pmatrix}. \tag{2}$$

Analogously, the matrix B does not have a totally non-positive completion since if $detA$ $[\{1,2\}|\{2,4\}] \leq 0$, then $x_{24} \leq 2$, and if $detA[\{1,2\}|\{1,3\}] \leq 0$, then $x_{24} \geq 3$, which is impossible.

The specified positions in an $n \times n$ partial matrix $A = (a_{ij})$ can be represented by a graph $G_A = (V, E)$, where the set of vertices V is $\{1, 2, \cdots, n\}$, $i \neq j$, is an edge or arc if the (i, j) entry is specified. G_A is an undirected graph when A is combinatorially symmetric and a directed