J. At. Mol. Sci.
doi: 10.4208/jams.062511.081411a

Vol. **3**, No. 2, pp. 136-151
May 2012

The new multi-order exact solutions of some nonlinear evolution equations

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Received 25 June 2011; Accepted (in revised version) 14 August 2011
Published Online 8 November 2011

Abstract. Based on the Lamé equation and Jacobi elliptic function, the perturbation method is applied to some nonlinear evolution equations. And there many multi-order solutions are derived to these nonlinear evolution equations. These multi-order solutions correspond to the different periodic solutions, which can degenerate to the different soliton solutions. The method can be also applied to many other nonlinear evolution equations.

PACS: 04.30.Nk, 02.90.+p

Key words: Lamé equation, Lamé function, multi-order exact solutions, Jacobi elliptic function, perturbation method, nonlinear evolution equations

1 Introduction

To find the exact solutions of the nonlinear evolution equations plays an important role in nonlinear studies. Applying some new methods, such as inverse scattering transformation [1], Bäcklund transformation [2], Darboux transformation [3], Hirota method [4], homogeneous balance method [5], Lie group method [6], sine-cosine method [7], homotopy perturbation method [8], variational method [9], tanh method [10], exp-method [11] and the JEFE method [12,13] and so on, many exact solutions are obtained, from which rich structures are shown to exist in different nonlinear wave equations. Furthermore, in order to discuss the stability of these solutions, one must superimpose a small disturbance on these solutions and analysis the evolution of the small disturbance. This is equivalent to the solutions of nonlinear evolution equations expanded as a power series in terms of a small parameter and derive multi-order exact solutions [14-17]. In the paper, using Jacobi elliptic function expansion method, the new multi-order periodic solutions of four nonlinear evolution equations are obtained by means of the Jacobi elliptic function and the new Lamé functions. They contain

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some previous exact periodic solutions. At the limit condition, the periodic solutions give corresponding solitary wave solutions.

This paper is organized as follows: In Section 2, we give the introduction of the Lamé function. In Section 3 and Section 4, we apply two kinds of Lamé functions $L_2(\xi)$ and $L_3(\xi)$ to solve nonlinear evolution equations and to derive their corresponding multi-order exact solutions, respectively. Conclusions will be presented in finally.

2 Lamé function

In general, the Lamé equation [18,19] in terms of $y(x)$ can be written as

$$\frac{d^2y(x)}{dx^2}+\left[\lambda-p(p+1)\mathrm{cs}^2(\xi)\right]y(x)=0, \tag{1}$$

where λ is an eigenvalue, p is a positive integer, $\mathrm{cs}(\xi)=\mathrm{cn}(\xi)/\mathrm{sn}(\xi)$ is a kind of Jacobi elliptic functions with its modulus m $(0<m<1)$.

Set

$$\eta=\mathrm{cs}^2(x). \tag{2}$$

Then the Lamé equation (1) becomes

$$\frac{d^2y}{d\eta^2}+\frac{1}{2}\left(\frac{1}{\eta}+\frac{1}{\eta+1}+\frac{h}{h\eta+h-1}\right)\frac{dy}{d\eta}-\frac{\mu+p(p+1)\eta h}{4\eta(\eta+1)(h\eta+h-1)}y=0, \tag{3}$$

where

$$h=m^{-2}>1, \qquad \mu=-h\lambda \tag{4}$$

Eq. (3) is a kind of the Fuchs-typed equations with four regular points $\eta=0$, -1, $h^{-1}-1$ and ∞, the solution of the Lamé equation (1) is known as Lamé function.

For example, when $p=2$, $\lambda=m^2-2$, i.e. $\mu=-h\lambda=-(1-2m^{-2})$, the Lamé function is

$$L_2(x)=(1-h^{-1}+\eta)^{\frac{1}{2}}(1+\eta)^{\frac{1}{2}}=\mathrm{ds}(x)\mathrm{ns}(x), \tag{5}$$

when $p=3$, $\lambda=4(m^2-2)$, i.e. $\mu=-h\lambda=-4(1-2m^{-2})$, the Lamé function is

$$L_3(x)=\eta^{\frac{1}{2}}(1-h^{-1}+\eta)^{\frac{1}{2}}(1+\eta)^{\frac{1}{2}}=\mathrm{cs}(x)\mathrm{ds}(x)\mathrm{ns}(x). \tag{6}$$

In (5) and (6), $\mathrm{ns}(x)=1/\mathrm{sn}(x)$ and $\mathrm{ds}(x)=\mathrm{dn}(x)/\mathrm{sn}(x)$ are two kinds of the Jacobi elliptic functions. In the next sections, we will apply these two kinds of Lamé functions $L_2(x)$ and $L_3(x)$ to solve nonlinear evolution equations and to derive their corresponding multi-order exact solutions.