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LOW-RANK TENSOR STRUCTURE OF SOLUTIONS TO ELLIPTIC PROBLEMS WITH JUMPING COEFFICIENTS*

Sergey Dolgov
Moscow Institute of Physics and Technology, Russia
Email: sergey.v.dolgov@gmail.com
Boris N. Khoromskij
Max-Planck-Institute for Mathematics in Sciences, Inselstr. 22-26, D-04103 Leipzig, Germany
Email: bokh@mis.mpg.de
Ivan Oseledets
Institute of Numerical Mathematics, Russian Academy of Sciences,
Gubkina 8, 119991 Moscow, Russia
ivan.oseledets@gmail.com
Eugene E. Tyrtyshnikov
Institute of Numerical Mathematics, Russian Academy of Sciences,
Gubkina 8, 119991 Moscow, Russia;
Lomonosov Moscow State University, Russia; University of Siedlce, Poland (visiting professor)
Email: tee@inm.ras.ru

Abstract

We study the separability properties of solutions to elliptic equations with piecewise constant coefficients in $\mathbb{R}^d$, $d \geq 2$. The separation rank of the solution to diffusion equation with variable coefficients is presented.

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1. Introduction

In this paper, we study the separability properties of solutions to elliptic equations with piecewise constant coefficients. By a separable decomposition of a multivariate function, we mean its representation or approximation by a sum of the products of univariate functions. The separability properties of the Laplace operator inverse and hence of the solution to Poisson equation were estimated in [1–4]. In what following, a point to study is the dependence on structure of the diffusion coefficient.

To fix the idea, we first consider a model elliptic boundary value problem in two dimensions,

$$-\nabla(a\nabla u) = f, \quad \text{in} \quad \Omega = [0,1]^2, \tag{1.1a}$$

$$u|_{\partial\Omega} = 0, \tag{1.1b}$$

with an assumption that f is represented by a piecewise smooth tensor decomposition

$$f(x,y) = \sum_{k=1}^{r_f} f_k^{(1)}(x)\ f_k^{(2)}(y), \tag{1.2}$$

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and the diffusion coefficient $a(x, y)$ is a piecewise constant function on cells of a tensor grid in Ω. In the case of an $M \times M$ tensor tiling, the reciprocals $1/a$ on these cells comprise a matrix of the form

$$B = \begin{bmatrix} 1/a_{11} & \cdots & 1/a_{1M} \\ \vdots & \ddots & \vdots \\ 1/a_{M1} & \cdots & 1/a_{MM} \end{bmatrix} \tag{1.3}$$

with the notation

$$r_{1/a} = \mathrm{rank} B.$$

Clearly, the function $1/a$ has the same separable form,

$$1/a(x, y) = \sum_{l=1}^{r_{1/a}} b_l^{(1)}(x) \cdot b_l^{(2)}(y) = \sum_{l=1}^{r_{1/a}} \frac{1}{a_l^{(1)}(x)} \cdot \frac{1}{a_l^{(2)}(y)}, \tag{1.4}$$

which can be shown by a constant spline interpolation. Given $\varepsilon > 0$, we approximate u by a separable decomposition

$$u_{r_u} = \sum_{k=1}^{r_u} u_k^{(1)}(x)\, u_k^{(2)}(y), \tag{1.5}$$

so that $\|u - u_{r_u}\|_{L^\infty} \leq \varepsilon$.

In this paper we investigate how r_u depends on ε, $r_{1/a}$, M and r_f. Straightforward analysis in the continuous case gives the following rank estimation,

$$r_u = \mathcal{O}(M^2 r_v),$$

where r_v is the maximal ε-rank of the solution in each domain generated by the $M \times M$ tiling. Notice that r_v depends weakly on a, since in each domain the solution satisfies just the Poisson equation: $-a\Delta u = f$.

In the 3D or higher dimensional case we formulate the problem in a similar way. Consider

$$-\nabla(a\nabla u) = f, \quad \text{in} \quad \Omega = [0, 1]^d, \tag{1.6a}$$

$$u|_{\partial\Omega} = 0, \tag{1.6b}$$

and assume a separability property for the right-hand side,

$$f(\mathbf{x}) = \sum_{k=1}^{r_f} f_k^{(1)}(x_1) \cdots f_k^{(d)}(x_d), \tag{1.7}$$

and the reciprocal diffusion coefficient,

$$1/a(\mathbf{x}) = \sum_{l=1}^{r_{1/a}} b_l^{(1)}(x_1) \cdots b_l^{(d)}(x_d) = \sum_{l=1}^{r_{1/a}} \frac{1}{a_l^{(1)}(x_1)} \cdots \frac{1}{a_l^{(d)}(x_d)}. \tag{1.8}$$

Now for given $\varepsilon > 0$, we approximate u by a separable decomposition

$$u_{r_u} = \sum_{k=1}^{r_u} u_k^{(1)}(x_1) \cdots u_k^{(d)}(x_d), \tag{1.9}$$

so that $\|u - u_{r_u}\|_{L^\infty} \leq \varepsilon$. Such a decomposition is crucial for the numerical solution of the problem. Suppose we discretize the problem on the grid with n points in each spatial direction.