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SOLVING THE BACKWARD HEAT CONDUCTION PROBLEM BY DATA FITTING WITH MULTIPLE REGULARIZING PARAMETERS*

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Abstract

We propose a new reconstruction scheme for the backward heat conduction problem. By using the eigenfunction expansions, this ill-posed problem is solved by an optimization problem, which is essentially a regularizing scheme for the noisy input data with both the number of truncation terms and the approximation accuracy for the final data as multiple regularizing parameters. The convergence rate analysis depending on the strategy of choosing regularizing parameters as well as the computational accuracy of eigenfunctions is given. Numerical implementations are presented to show the validity of this new scheme.

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Key words: Inverse problem, Data fitting, Regularization, Convergence rate, Numerics.

1. Introduction

For a bounded domain $\Omega \subset \mathbb{R}^N (N = 1, 2, 3)$, consider the heat conduction problem

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (a(x)\nabla u), & x \in \Omega, t > 0 \\ u(x,t) = 0, & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

For given initial data $u_0(x)$, this forward problem is well-posed(Chapter 3, Theorem 3.2, [13]), which defines a map $\mathcal{G} : u_0(\cdot) \in L^2(\Omega) \mapsto u(\cdot, T) \in H_0^1(\Omega)$.

Now assume that $u_0(x)$ is unknown, while the final data is given by $u(x,T) = f(x), x \in \Omega$. The backward problem is to solve $u(x,t)$ for $t \in [0,T)$ from given $f(x)$ or its measurement data $f^\delta(x)$ satisfying $\|f^\delta - f\|_{L^2(\Omega)} \le \delta$ for some known error level $\delta > 0$. It is well-known that this problem is ill-posed due to the irreversibility of heat conduction along time direction.

For this ill-posed problem with wide engineering background [19, 20], many regularizing schemes have been researched thoroughly, which focus on the construction of the approximate solution $u^\delta(x,t)$ from $f^\delta(x)$ and the convergence rate analysis on $\|u^\delta(\cdot,t) - u(\cdot,t)\|$ as $\delta \to 0$. Of course, these two issues depend on the regularizing scheme. One of the well-known scheme

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is the so-called quasi-reversibility method [3], which firstly constructs the regularizing solution $u_0^\delta(x)$ for the initial data and then gets $u^\delta(x,t)$ for $t \in (0,T)$ by solving the direct problem (1.1). The convergence of such kinds of schemes can be established in terms of the convergence of initial data $u_0^\delta(x)$, see [4–6, 9]. For solving $u^\delta(x,t)$ for $t \in (0,T)$ directly from $f^\delta(x)$ with the Hölder stability of order $\frac{t}{T}$, the readers are refereed to [1, 17, 18, 23]. The other work for backward heat problem can be found in [2, 15, 16, 21].

Recently, some attempts to construct a regularizing solution with explicit expression have received much attention. The advantage of this new idea is that the well-posedness of the regularizing problem is guaranteed automatically, provided that the noisy input data be modified appropriately. Then the numerical computation of the regularizing solution for all $t \in [0,T)$ is much easy, for example, see [4, 7, 14, 22] for the mollification method. We call such kind of scheme as data regularization.

In this paper, we propose a new regularizing scheme along this direction. By expanding the noisy data $f^\delta(x)$ in terms of the base functions $\{\varphi_k(x,T) : k \in \mathbb{N}\}$ solved from the heat conduction process, the regularizing data for the noisy measurement are constructed using the finite approximate terms expansion, where both the number of expansion terms and the approximate accuracy are considered as the regularizing parameters simultaneously. Then the regularizing solution $u^\delta(x,t)$ for all $t \in [0,T)$ can be constructed from the approximate final data explicitly. In this regularizing scheme for backward heat problem, all the ill-posedness is concentrated on the final data fitting process. Such a scheme is essentially a regularizing technique for the input data. We analyze the convergence of this new scheme and give some numerical implementations. It is interesting that our regularizing scheme provides the convergence rate of $\|u^\delta(\cdot,t) - u(\cdot,t)\|$ decreasing by the factor $e^{-\lambda_1 t}$ for fixed error level $\delta > 0$, which is physically reasonable from the smoothing property of direct heat conduction process, where $\lambda_1 > 0$ is the minimum eigenvalue of the operator $-\nabla \cdot (a(x)\nabla)$.

We would like to emphasize the difference between our data fitting technique and the classical TSVD method to deal with the linear ill-posed problems. For our problem, $u(x,t)$ for $t \in [0,T)$ satisfies a linear integral equation of the first kind, so the TSVD method can be used to solve this equation, where Tikhonov regularization can be combined together to determine the truncation term from the noise level. In this scheme, the regularization technique is applied at each time $t \in [0,T)$, and therefore the regularization equation should be solved for every time t. However, in our data fitting scheme, we only regularize the final measurement data $u^\delta(x,T)$ by its base function expansion, with both the truncation term and the approximate accuracy as regularizing parameters. Then the approximate solution for any $t \in [0,T)$ can be expressed explicitly using the spatial base function of elliptic operator. In other words, we extract the ill-posedness of the problem from the original parabolic system with the help of the eigensystem of elliptic operator $-\nabla \cdot (a(x)\nabla u)$. Therefore, the novelty of the proposed scheme in this paper compared with the classical TSVD method is that we can decrease the amount of computations by solving the regularizing equation only one times at $t = T$ and then get the regularizing solution for all $t \in [0,T)$ explicitly with convergence rate estimate. Moreover, we also analyze the influence of the computational error for the eigensystem and give an explicit error estimate.

This paper is organized as follows. In Section 2, we construct the regularizing solution explicitly. Then in Section 3, we give the convergence analysis on the regularizing solution using the exact eigenfunction expansions. In Section 4, we consider the convergence for the noisy eigensystem, noticing that both the eigenfunctions and the eigenvalues must be computed numerically for general heat conduction system. In this case, the error η in computing the