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ON THE INFLUENCE OF THE WAVENUMBER ON COMPRESSION IN A WAVELET BOUNDARY ELEMENT METHOD FOR THE HELMHOLTZ EQUATION

STUART C. HAWKINS, KE CHEN, AND PAUL J. HARRIS

Abstract. We examine how the wavenumber influences the compression in a wavelet boundary element method for the Helmholtz equation. We show that for wavelets with high vanishing moments the number of nonzeros in the resulting compressed matrix is approximately proportional to the square of the wavenumber. When the wavenumber is fixed, the wavelet boundary element method has optimal complexity with respect to the number of unknowns. When the mesh spacing is proportional to the wavelength, the complexity of the wavelet boundary element method is approximately proportional to the square of the number of unknowns.

Key Words. wavelets, boundary element method, and Helmholtz equation.

1. Introduction

Techniques based on boundary integral equations (BIEs) are attractive for three dimensional partial differential equations (PDEs) because they are defined on a two dimensional surface rather than a three dimensional domain. Two dimensional problems can typically be solved with many fewer unknowns than three dimensional problems. BIE formulations are particularly attractive when the three dimensional domain is infinite, for example in an exterior problem.

Traditional boundary element methods (BEMs) for solving integral equations are very effective but have $O(N^2)$ complexity and require $O(N^2)$ storage, where N is the number of unknowns. There is much interest in fast solution techniques for BIEs. Prominent fast techniques include the fast multipole method (FMM) [16, 9], equivalent sources [4, 3], spectral basis functions [1, 8], and wavelets [2, 7, 15, 17, 14, 6].

The use of wavelets for the compression of operators was pioneered by Beylkin, Coifman and Rokhlin [2], who showed that operators of Calderon-Zygmund type can be represented, to a specified accuracy, by a sparse matrix. Dahmen, Prössdorf and Schneider [7] linked the accuracy to the discretisation error. Later developments by Rathsfeld [15], von Petersdorff and Schwab [17], and Lage and Schwab [14] yielded sophisticated discretisation schemes with (a) complexity $O(N \log^a N)$ for some small integer a, and (b) asymptotic order of convergence that is not changed by the compression. The wavelet Galerkin scheme of Dahmen, Harbrecht and Schneider [6] has the optimal complexity $O(N)$. For the Laplace problem these wavelet schemes have been successfully demonstrated. In contrast, for the Helmholtz problem wavelet schemes have not been widely used. The oscillatory nature of the kernel

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in BIEs for the Helmholtz equation is considered an obstacle to compression using wavelets.

Wagner and Chew [18] showed that their wavelet discretisation scheme for the related electromagnetic scattering problem yields a compressed matrix with $O(N^2)$ nonzeros. Thus wavelets are not useful in that context. However, in that study the number of unknowns was tied to the wavenumber k so that the number of nodes per wavelength remained constant. Huybrechs, Simoens and Vandewalle [13] showed that, for the two dimensional Helmholtz problem, the wavelet Galerkin scheme has approximate complexity $O(kN)$. When the number of unknowns is tied to the wavenumber this becomes $O(N^2)$ and is in agreement with [18].

Here we consider application of the wavelet Galerkin scheme in [6, 10] for the three dimensional Helmholtz problem. We analyse the wavenumber dependence of the compression obtained with the wavelets so that we can assess how useful the wavelet discretisation scheme is in this context. We explicitly consider several integral operators, including the single and double layer potentials and the hypersingular operator. We show that there are approximately $O(k^2N)$ nonzeros in the compressed matrix. The exact exponent of k depends on the order of the integral operator considered and on the vanishing moments of the wavelets.

Practitioners might increase the number of unknowns N to (a) maintain the number of nodes per wavelength as k increases; or (b) increase the accuracy of the solution for fixed k. We show that in case (a) the wavelet Galerkin scheme has approximate complexity $O(N^2)$, which is in agreement with Wagner and Chew [18]. In contrast, in case (b) the wavelet Galerkin scheme has optimal complexity $O(N)$, which is the complexity observed for the Laplace problem [6, 10].

We write $a(k,N) = O(b(k,N))$ when there exist constants $C, M \geq 0$ such that $a(k,N) \leq Cb(k,N)$ for all $N > M$. We write $a(k,N) \lesssim b(k,N)$ when there exists constant $C > 0$ such that $a(k,N) \leq Cb(k,N)$ for all N. We write $a(k,N) \sim b(k,N)$ when $a(k,N) \lesssim b(k,N)$ and $b(k,N) \lesssim a(k,N)$.

In Section 2 we state the Laplace and Helmholtz problems and review the relevant boundary integral operators for these problems. In Section 3 we review the relevant details of the wavelet Galerkin scheme [6, 10]. In Section 4 we review the compression theory for the wavelet Galerkin scheme applied to the Laplace problem. In Section 5 we state the compression theory for the wavelet Galerkin scheme applied to the Helmholtz problem. In Section 6 we present numerical results.

2. Integral Operators and Equations

We consider here the exterior Laplace and Helmholtz problems. Both problems can be reformulated as boundary integral equations and we list the component boundary integral operators. The treatment for the interior problem is similar, and the same boundary integral operators appear.

Let $D \subseteq \mathbb{R}^3$ be a bounded region with surface Γ and exterior $D_+ = \mathbb{R}^3 \setminus (D \cup \Gamma)$. The exterior Laplace equation is

$$\nabla^2 u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in D_+ \cup \Gamma. \tag{1}$$

The single and double layer potentials for the Laplace equation are

$$(\mathcal{L}u)(\boldsymbol{x}) = \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})ds_{\boldsymbol{y}}, \quad (\mathcal{M}u)(\boldsymbol{x}) = \int_\Gamma \frac{\partial G}{\partial \boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})ds_{\boldsymbol{y}},$$

where $G(\boldsymbol{x},\boldsymbol{y}) = 1/4\pi\|\boldsymbol{x}-\boldsymbol{y}\|$ is the free space Green's function for the Laplace equation.