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ON NEWTON-HSS METHODS FOR SYSTEMS OF NONLINEAR EQUATIONS WITH POSITIVE-DEFINITE JACOBIAN MATRICES*

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Abstract

The Hermitian and skew-Hermitian splitting (HSS) method is an unconditionally convergent iteration method for solving large sparse non-Hermitian positive definite system of linear equations. By making use of the HSS iteration as the inner solver for the Newton method, we establish a class of Newton-HSS methods for solving large sparse systems of nonlinear equations with positive definite Jacobian matrices at the solution points. For this class of inexact Newton methods, two types of local convergence theorems are proved under proper conditions, and numerical results are given to examine their feasibility and effectiveness. In addition, the advantages of the Newton-HSS methods over the Newton-USOR, the Newton-GMRES and the Newton-GCG methods are shown through solving systems of nonlinear equations arising from the finite difference discretization of a two-dimensional convection-diffusion equation perturbed by a nonlinear term. The numerical implementations also show that as preconditioners for the Newton-GMRES and the Newton-GCG methods the HSS iteration outperforms the USOR iteration in both computing time and iteration step.

Mathematics subject classification: 65F10, 65W05.
Key words: Systems of nonlinear equations, HSS iteration method, Newton method, Local convergence.

1. Introduction

Large sparse systems of nonlinear equations arise in many areas of scientific computing and engineering applications, e.g., in discretizations of nonlinear differential and integral equations, numerical optimization and so on; see [10, 26, 27] and references therein.

Let $F : \mathbb{D} \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a nonlinear and continuously differentiable mapping defined on the open convex domain $\mathbb{D}$ in the n-dimensional complex linear space $\mathbb{C}^n$, and consider systems of nonlinear equations of the form

$$F(x) = 0. \tag{1.1}$$

We assume that the Jacobian matrix of the nonlinear function $F(x)$ at the solution point $x_\star \in \mathbb{D}$, denoted as $F'(x_\star)$, is sparse, non-Hermitian, and positive definite. Here, the matrix $F'(x)$, for

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$x \in \mathbb{D}$, is said to be positive definite if its Hermitian part

$$\mathcal{H}(F'(x)) := \frac{1}{2}(F'(x) + F'(x)^*)$$

is positive definite, where $F'(x)^*$ represents the conjugate transpose of $F'(x)$. For notational convenience, we also denote by

$$\mathcal{S}(F'(x)) := \frac{1}{2}(F'(x) - F'(x)^*)$$

the skew-Hermitian part of $F'(x)$; see [7, 12, 15, 18]. In this paper, we will study effective iteration methods and their convergence properties for solving this class of nonlinear systems.

The most classic and important solver for the system of nonlinear equations (1.1) may be the Newton method, which can be formulated as

$$x^{(k+1)} = x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}), \qquad k = 0, 1, 2, \ldots, \tag{1.2}$$

where $x^{(0)} \in \mathbb{D}$ is a given initial vector; see [11,26,27,29]. Obviously, at the k-th iteration step we need to solve the so-called Newton equation

$$F'(x^{(k)})s^{(k)} = -F(x^{(k)}), \quad \text{with} \quad x^{(k+1)} := x^{(k)} + s^{(k)}, \tag{1.3}$$

which is the dominant task in implementations of the Newton method. When the Jacobian matrix $F'(x)$ is large and sparse, iterative methods either of the splitting relaxation form (e.g., Gauss-Seidel, SOR [1)] and USOR [2)] ; see [19, 26]) or of the Krylov subspace form (e.g., GMRES, BiCGSTAB and GCG [3)] ; see [4,25,28]) are often the methods of choice for effectively computing an approximation to the update vector $s^{(k)}$; see also [1, 2, 5, 6, 13]. This naturally results in the following inexact version of the Newton method for solving the system of nonlinear equations (1.1):

$$x^{(k+1)} = x^{(k)} + s^{(k)}, \quad \text{with} \quad F'(x^{(k)})s^{(k)} = -F(x^{(k)}) + r^{(k)}, \tag{1.4}$$

where $r^{(k)}$ is a residual yielded by the inner iteration due to the inexact solving; see [10,11,21,23]. Note that the convergence of the splitting relaxation methods is guaranteed only for Hermitian positive definite matrices or H-matrices, while this class of methods often requires much less computing operations at each iteration step and also much less computer storage than the Krylov subspace methods in actual implementations.

Recently, a Hermitian and skew-Hermitian splitting (HSS) iteration method was presented in [15] for solving large sparse system of linear equations with a non-Hermitian positive definite coefficient matrix, say $A \in \mathbb{C}^{n\times n}$; see also [12, 18]. Theoretical analysis has demonstrated that the HSS iteration method converges unconditionally to the exact solution, with the bound on the rate of convergence about the same as that of the conjugate gradient method when applied to the Hermitian matrix $\mathcal{H}(A) := \frac{1}{2}(A+A^*)$, and numerical experiments have shown that the HSS iteration method is very efficient and robust for solving non-Hermitian positive definite linear systems. Moreover, the HSS iteration method possesses a comparative memory requirement, but faster convergence rate, than the USOR iteration method, especially for matrices having strong skew-Hermitian parts.

[1)] SOR represents the successive overrelaxation method.
[2)] USOR represents the unsymmetric successive overrelaxation method.
[3)] GCG represents the generalized conjugate gradient method.