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WEIGHTED L^2-NORM A POSTERIORI ERROR ESTIMATION OF FEM IN POLYGONS

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Abstract. In this paper, we generalize well-known results for the L^2-norm a posteriori error estimation of finite element methods applied to linear elliptic problems in convex polygonal domains to the case where the polygons are non-convex. An important factor in our analysis is the investigation of a suitable dual problem whose solution, due to the non-convexity of the domain, may exhibit corner singularities. In order to describe this singular behavior of the dual solution certain weighted Sobolev spaces are employed. Based on this framework, upper and lower a posteriori error estimates in weighted L^2-norms are derived. Furthermore, the performance of the proposed error estimators is illustrated with a series of numerical experiments.

Key Words. Finite element methods, a posteriori error analysis, L^2-norm error estimation, non-convex polygonal domains.

1. Introduction

Given a (possibly non-convex) bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with (Lipschitz) boundary $\Gamma = \partial\Omega$, and a function $f \in L^2(\Omega)$, we consider the elliptic model problem

$$-\Delta u = f \qquad \text{in } \Omega \tag{1}$$

$$u = 0 \qquad \text{on } \Gamma. \tag{2}$$

The standard weak formulation of (1)–(2) reads: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, d\boldsymbol{x} = \int_\Omega f v \, d\boldsymbol{x} \tag{3}$$

for all $v \in H_0^1(\Omega)$. Here and it what follows, we use the following notation: For a domain $D \subset \mathbb{R}^n$ ($n = 1$ or $n = 2$) and an integer $k \in \mathbb{N}_0$, we denote by $H^k(D)$ the usual Sobolev space of order k on D, with norm $\|\cdot\|_{k,D}$ and semi-norm $|\cdot|_{k,D}$. The space $H_0^1(\Omega)$ is defined as the subspace of $H^1(\Omega)$ with zero trace on $\partial\Omega$. Furthermore, $H^{-1}(D)$ denotes the dual space of $H_0^1(D)$, and $L^2(D) = H^0(D)$.

In order to discretize the variational formulation (3) by a finite element method, we consider a regular subdivision $\mathcal{T}_{FE}$ (finite element mesh) of Ω into disjoint open triangles K (elements), i.e. $\mathcal{T}_{FE} = \{K\}$, $\bigcup_{K\in\mathcal{T}_{FE}} \overline{K} = \overline{\Omega}$. By h_K, we denote the diameter of an element $K \in \mathcal{T}_{FE}$. We assume that the subdivision $\mathcal{T}_{FE}$ is shape-regular (see, e.g., [6]) and of local bounded variation. The latter assumption means

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that there exists a constant $\sigma > 1$ such that $\sigma^{-1} < h_{K_\sharp}/h_{K_\flat} < \sigma$, for any two neighboring elements $K_\sharp$ and $K_\flat$. Moreover, we introduce the finite element space

$$\mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE}) = \{\phi \in H_0^1(\Omega) : \phi|_K \in \mathcal{P}_1(K), K \in \mathcal{T}_{FE}\},$$

where, for $k \in \mathbb{N}_0$ and $K \in \mathcal{T}_{FE}$, $\mathcal{P}_k(K)$ is defined as the set of all polynomials of total degree (at most) k on K.

A finite element approximation of the exact solution $u \in H_0^1(\Omega)$ of (1)–(2) can now be obtained in the usual way by finding the unique solution $u_{FE} \in \mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$ of the discrete variational formulation

$$a(u_{FE}, v) = \ell(v) \qquad \forall v \in \mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE}), \tag{4}$$

where

$$a(w, v) = \int_\Omega \nabla w \cdot \nabla v \, d\boldsymbol{x}, \qquad \ell(v) = \int_\Omega f v \, d\boldsymbol{x}.$$

Clearly, there holds the Galerkin orthogonality

$$a(e_{FE}, v) = 0 \qquad \forall v \in \mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE}). \tag{5}$$

Here, e_{FE} is the finite element error given by

$$e_{FE} = u - u_{FE}, \tag{6}$$

where $u \in H_0^1(\Omega)$ is again the exact solution of (1)–(2), and $u_{FE} \in \mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$ is its numerical approximation from (4).

Standard techniques for the a posteriori error estimation of the L^2-norm of e_{FE}, consist usually of the following two main steps (Aubin-Nitsche trick; see, e.g., [1, 5, 12], and the references therein): Firstly, a suitable dual problem is formulated; this makes it possible to relate the L^2-norm of e_{FE} to the finite element method (4). Secondly, using the Galerkin orthogonality (5), the L^2-error $\|e_{FE}\|_{L^2(\Omega)}$ is bounded in terms of some approximation errors between the solution of the dual problem and an appropriate interpolant in the finite element space $\mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$. Here, standard L^2-norm a posteriori analyses are typically based on approximation results that require the global H^2-regularity of the dual problem, which is in fact available if the polygonal domain Ω is convex; see [2, 7, 8], for example. In non-convex polygons, however, this assumption does generally not hold; here, due to the presence of corner singularities, the regularity of the dual problem is typically reduced to $H^{1+\varepsilon}$, for $\varepsilon < 1$. Consequently, standard H^2-approximation results cannot be applied.

The goal of this paper is to generalize the above-mentioned approach for the L^2-norm a posteriori error estimation of the finite element error e_{FE} to the case where the domain Ω is a possibly non-convex polygon. To this end, we describe the regularity of the dual problem in terms of weighted Sobolev spaces using the results in [2], and apply some appropriate interpolation results (see, e.g., [11], and the references therein) to approximate its solution in the finite element space $\mathcal{S}_0^{1,1}(\Omega, \mathcal{T}_{FE})$. We will then be able to derive upper a posteriori bounds for some weighted L^2-norms of the error e_{FE}; see Theorem 3.1. More precisely, we will show that there holds an estimate of the form

$$\|\Phi^{-1} e_{FE}\|_{L^2(\Omega)}^2 \leq C \sum_{K \in \mathcal{T}_{FE}} \eta_K(u_{FE}, f)^2, \tag{7}$$

where Φ is a certain weight function (associated to Ω), and η_K, $K \in \mathcal{T}_{FE}$, are local error indicators depending on the mesh $\mathcal{T}_{FE}$, the finite element solution u_{FE} from (4) and the right-hand side f in (1). In addition, using suitable cut-off functions (see [9]), we will also prove some (weighted) local lower bounds; see Theorem 3.4.