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CONVERGENCE RATES FOR DIFFERENCE SCHEMES FOR POLYHEDRAL NONLINEAR PARABOLIC EQUATIONS*

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Abstract

We build finite difference schemes for a class of fully nonlinear parabolic equations. The schemes are polyhedral and grid aligned. While this is a restrictive class of schemes, a wide class of equations are well approximated by equations from this class. For regular ($C^{2,\alpha}$) solutions of uniformly parabolic equations, we also establish of convergence rate of $\mathcal{O}(\alpha)$. A case study along with supporting numerical results is included.

Mathematics subject classification: 65M06, 65M12.
Key words: Error estimates, Convergence rate, Viscosity solutions, Finite difference schemes

1. Introduction

Although the theory of viscosity solutions has been well established for a broad class of nonlinear elliptic and parabolic equations, there are no general methods available for building convergent difference schemes to solve these equations. Schemes need to be custom built for each equation, or for classes of equations.

For degenerate, quasilinear equations such as motion of level sets by mean curvature, and the infinity Laplace equations, specialized convergent schemes have been build [13,14]. Convergent schemes have been built for the class of equations which are functions of the eigenvalues of the Hessian [16]. In general, these schemes requires successively wider stencils in order to converge. This means that the approximation error depends on an additional parameter, $d\theta$, the directional resolution. In practice, schemes of width one or two are sufficient, since the $d\theta$ error is small compared to the spatial resolution error.

In this article, we focus on the particular subclass of *polyhedral grid aligned* equations. The subclass is artificial: it is designed for the purpose of building convergent schemes. However, many of the previously mentioned equations can be approximated by this class. We build convergent schemes, and establish error estimates, which depend on the regularity of the solutions.

Related results

Convergence rates for second order elliptic and parabolic equations, without any regularity assumptions, are obtained in example Krylov [8,11], Kuo and Trudinger [12], Barles and Jacobsen [1], and Caffarelli and Souganidis [5] and the references therein. The methods used come from regularity theory for nonlinear elliptic PDEs and are substantially more technical than the methods herein.

Here we obtain convergence rates using available regularity results. This approach simplifies the argument considerably, since it avoids a reiteration of the regularity theory.

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Contents

The remainder of this section recalls the setting for our nonlinear parabolic equations and the necessary regularity results.

Section 2 is a case study with a simple example equation. Error estimates are obtained directly in this simpler setting, and supporting numerical results are presented.

The first part of section 3 recalls general results on nonlinear elliptic schemes. The second part presents new material on error estimates in terms of the residual for perturbed equations, the methods of lines, and finally for fully discrete difference schemes.

The main results are in the section 4. Here the class of schemes is established. The schemes are shown to be elliptic, and consistent, which is enough to prove convergence. Then the error estimates of the previous section are used to obtain a convergence rate.

1.1. Nonlinear parabolic equations

Our results concern the fully nonlinear parabolic Partial Differential Equation (PDE)

$$u_t(x,t) + F[u](x,t) = 0, \qquad \text{for } (x,t) \text{ in } \Omega \times [0,T) \tag{PDE}$$

where Ω is a domain in $\mathbb{R}^n$, along with initial and boundary conditions

$$\begin{cases} u(x,t) = g(x,t), & \text{for } (x,t) \text{ on } \Omega \times \{0\} \\ u(x,t) = h(x,t), & \text{for } (x,t) \text{ on } \partial\Omega \times (0,T). \end{cases} \tag{BC}$$

The fully nonlinear elliptic partial differential operator $F[u]$ is given by

$$F[u](x) \equiv F(D^2u(x), Du(x), u(x), x). \tag{1.1}$$

Here Du and D^2u denote the gradient and Hessian of u, respectively. The function $F(X,p,r,x)$ is defined on $\mathbb{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega$, and $\mathbb{S}^n$ is the space of symmetric $n \times n$ matrices. The natural setting for equations of this type is viscosity solutions [7].

Definition 1.1. *The differential operator* (1.1) *is* nonlinear *or* degenerate *elliptic if*

$$F(N,p,r,x) \le F(M,p,s,x) \text{ whenever } r \le s \text{ and } M \le N. \tag{1.2}$$

Here $M \le N$ means that $M - N$ is a nonnegative definite symmetric matrix. The corresponding parabolic operator (PDE) *is called* nonlinear *or* degenerate *parabolic.*

1.2. Regularity

When the equation F is convex and uniformly parabolic, solutions of (PDE) are $C^{2,\alpha}$, [17,18]. These results build upon the elliptic regularity [4,9]. In two dimensions and for special nonconvex equations, more regularity is available [3].

Here we use the convention of [10], where $C^{2,\alpha}$ means $C^{2,\alpha}$ in x and $C^{1,\alpha/2}$ in t.

Remark 1.1. It is often the case (for both theory and numerics) that the time variable scales quadratically with the space variable, as in [6] below.