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PARALLEL AUXILIARY SPACE AMG FOR $\boldsymbol{H}$(curl) PROBLEMS *

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Abstract

In this paper we review a number of auxiliary space based preconditioners for the second order definite and semi-definite Maxwell problems discretized with the lowest order Nédélec finite elements. We discuss the parallel implementation of the most promising of these methods, the ones derived from the recent Hiptmair-Xu (HX) auxiliary space decomposition [Hiptmair and Xu, SIAM J. Numer. Anal., 45 (2007), pp. 2483-2509]. An extensive set of numerical experiments demonstrate the scalability of our implementation on large-scale $\boldsymbol{H}$(curl) problems.

Mathematics subject classification: 65F10, 65N30, 65N55.
Key words: Parallel algebraic multigrid, H(curl) problems, Edge elements, Auxiliary space preconditioning.

1. Introduction

The numerical solution of electromagnetic problems based on Maxwell's equations is of critical importance in a number of engineering and physics applications. Recently, several implicit electromagnetic diffusion models have become popular in large-scale simulation codes [5, 21, 22]. These models require the solution of linear systems derived from discretizations of the weighted bilinear form

$$a(\boldsymbol{u},\boldsymbol{v}) = (\alpha\,\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v}) + (\beta\,\boldsymbol{u},\boldsymbol{v})\,, \tag{1.1}$$

where $\alpha > 0$ and $\beta \geq 0$ are piecewise-constant scalar coefficients describing the magnetic and electric properties of the medium. Problems involving (1.1) with $\beta > 0$ typically arise in time-domain electromagnetic simulations and are commonly referred to as second-order *definite* Maxwell equations. When β is zero in part of the domain, we call the problem *semi-definite*. Note that in this case the matrix corresponding to the bilinear form $a(\cdot,\cdot)$ is singular, and the right-hand side should satisfy appropriate compatibility conditions. One important semi-definite application is magnetostatics with a vector potential, where $\beta = 0$ in the whole domain.

Motivated by the needs of large multi-physics production codes, we are generally interested in efficient solvers for complicated systems of partial differential equations (PDEs), and in particular in the definite and semi-definite Maxwell problems. The target simulation codes run on parallel machines with tens of thousands of processors and typically lack discretization flexibility. Our focus is therefore on parallel algebraic methods since they can take advantage of needed discretization information about the problem only at the highest resolution.

If we restrict (1.1) to gradient fields, the familiar Poisson bilinear form is obtained, i.e.,

$$(\beta\,\nabla\phi,\nabla\psi) = a(\nabla\phi,\nabla\psi).$$

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One very general algebraic approach, which has had a lot of success on Poisson and more general (scalar and vector) elliptic bilinear forms is the family of Algebraic Multigrid (AMG) methods. Classical AMG couples a simple relaxation scheme with a hierarchy of algebraically constructed coarse-grid problems. Parallel implementations of AMG have been under intensive research and development in the last decade, and several scalable software libraries are currently available [8,10]. Unlike Poisson problems, however, it is well known that the Maxwell system has a large near-nullspace of gradients which has to be addressed explicitly by the solver. Therefore, a straightforward application of AMG (developed for elliptic problems) fails. For a theoretical explanation of this phenomena, see, e.g., [27].

When a hierarchy of structured meshes is available and proper finite elements are used, geometric multigrid can be successfully applied to (1.1) based on additional discretization information [11]. One way to take advantage of this fact is to extend AMG by incorporating algebraic versions of the Hiptmair smoothers used in the geometric case. This was the approach taken in [4, 14, 20]. Alternatively, one can try to reduce the original unstructured problem to a problem for the same bilinear form on a structured mesh, which can then be handled by geometric multigrid. This idea, known as the *auxiliary mesh* approach, was explored in [12,15].

Recently, Hiptmair and Xu proposed in [13] an auxiliary space preconditioner for (1.1) and analyzed it in the case of constant coefficients. This approach is computationally more attractive since, in contrast to the auxiliary mesh idea, it uses a nodal conforming auxiliary space on the original mesh. This property significantly simplifies the computations and, more importantly, allows us to harness the power of AMG for Poisson problems. A related method, which arrives at the same nodal subspaces based on a compatible discretization principle, was introduced in [6].

In this paper, we examine the connections between the above algorithms, and their application to large definite and semi-definite Maxwell problems. We concentrate on the lowest order Nédélec finite element discretizations and consider topologically complicated domains, as well as variable coefficients (including some that lead to a singular matrix).

The remainder of the document is structured as follows: in Section 2, we introduce the notation and summarize some basic facts concerning Nédélec finite element discretizations. In Section 3, we show that the methods from [13], [15] and [6] can be derived from a single decomposition of the Nédélec finite element space by introducing one or several additional (auxiliary) spaces. The algebraic construction of auxiliary space preconditioners is reviewed in Section 4, and our parallel implementation of the Hiptmair-Xu (HX) auxiliary (nodal) space based preconditioner is discussed in Section 5. Next, we present in Section 6 a number of computational experiments that demonstrate the scalability of this approach. We finish with some conclusions in the last section.

2. Notation and Preliminaries

We consider definite and semi-definite Maxwell problems based on (1.1) and posed on a fixed three-dimensional polyhedral domain Ω. Let $\mathcal{T}_h$ be a tetrahedral or hexahedral meshing of the domain which is globally quasi-uniform of mesh size h. For generality, we allow internal holes and tunnels in the geometry, i.e., Ω may have a multiply-connected boundary and need not be simply connected. Meshes with such features arise naturally in many practical applications.

Let $\boldsymbol{L}_2(\Omega)$, $\boldsymbol{H}_0^1(\Omega)$, $\boldsymbol{H}_0(\Omega;\mathrm{curl})$ and $\boldsymbol{H}(\Omega;\mathrm{div})$ be the standard Hilbert spaces corresponding to our computational domain, with respective norms $\|\cdot\|_0$, $\|\cdot\|_1$, $\|\cdot\|_{\mathrm{curl}}$ and $\|\cdot\|_{\mathrm{div}}$. Here,