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LOCAL A PRIORI AND A POSTERIORI ERROR ESTIMATE OF TQC9 ELEMENT FOR THE BIHARMONIC EQUATION*

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Abstract

In this paper, local a priori, local a posteriori and global a posteriori error estimates are obtained for TQC9 element for the biharmonic equation. An adaptive algorithm is given based on the a posteriori error estimates.

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Key words: Finite element, Biharmonic equation, A priori error estimate, A posteriori error estimate, TQC9 element.

1. Introduction

For a posteriori error estimates of finite elements, there has been a great deal of work (see, e.g., [1–5, 9, 14] and references therein). Most of the finite elements considered are mainly for the second-order partial differential equations. In the recent paper [12], local a priori and a posteriori error estimates of conforming and nonconforming elements for the biharmonic equation were discussed. In this paper, we consider the TQC9 element for the biharmonic equation.

The TQC9 (9-parameter quasi-conforming triangle) element was proposed by Tang et al. [6, 8] for the biharmonic equation. The TQC9 element also uses the degrees of freedom of the Zienkiewicz element, but unlike the Zienkiewicz element, it is convergent. The convergence property and a global a priori error estimate of the TQC9 element were proved in [10, 15, 16]. Here we will show local a priori, local a posteriori and global a posteriori error estimates of the TQC9 element.

Let $\Omega \subset R^2$ be a bounded polygonal domain with boundary $\partial\Omega$. For $f \in L^2(\Omega)$, we consider the homogeneous Dirichlet boundary value problem of the biharmonic equation:

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\nu = (\nu_1, \nu_2)^\top$ is the unit outer normal of $\partial\Omega$ and Δ is the standard Laplace operator.

Given a bounded domain $B \subset R^2$ and an integer m, let $H^m(B)$, $H_0^m(B)$, $\|\cdot\|_{m,B}$ and $|\cdot|_{m,B}$ denote the Sobolev space, the closure of $C_0^\infty(B)$ in $H^m(B)$, the corresponding Sobolev norm and semi-norm respectively. Let $H^{-m}(\Omega)$ denote the dual space of $H_0^m(\Omega)$ with norm $\|\cdot\|_{-m,\Omega}$.

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Let $i,j \in \{1,2\}$ and $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij} = \partial_i\partial_j$. For a function $v \in H^2(\Omega)$, we define

$$E(v) = \big(\partial_{11}v, \partial_{22}v, \partial_{12}v\big)^\top. \tag{1.2}$$

Let $\sigma \in [0, \frac{1}{2}]$ be the Poisson ratio and

$$K = \begin{pmatrix} 1 & \sigma & 0 \\ \sigma & 1 & 0 \\ 0 & 0 & 2(1-\sigma) \end{pmatrix}. \tag{1.3}$$

Define

$$a(v,w) = \int_\Omega E(w)^\top K E(v), \quad \forall v, w \in H^2(\Omega). \tag{1.4}$$

The weak form of problem (1.1) is: find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega), \tag{1.5}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

The TQC9 element for problem (1.5) and some known results will be given in Section 2. Section 3 will discuss local a priori error estimate of the TQC9 element. Section 4 will consider a posteriori error estimate. The last section gives some numerical results of an adaptive algorithm based on the a posteriori error estimate obtained.

2. TQC9 Element

Let (T, P_T, Φ_T) be the Zienkiewicz element with T a triangle, P_T the shape function space and Φ_T the set of nodal parameters consisting of the function values and two first order derivatives at three vertices of T (cf. [7]).

Let $\{\mathcal{T}_h(\Omega)\}$ be a family of shape regular triangulations by triangles with mesh size $h \to 0$. Let $h(x)$ be the function with its value the diameter h_T of the element T containing x.

Corresponding to $\mathcal{T}_h(\Omega)$, denote by $V_h(\Omega)$ and $V_{h0}(\Omega)$ the Zienkiewicz element spaces with respect to $H^2(\Omega)$ and $H_0^2(\Omega)$ respectively. It is known that $V_h(\Omega) \not\subset H^2(\Omega)$, $V_{h0}(\Omega) \not\subset H_0^2(\Omega)$, and $V_h(\Omega) \subset H^1(\Omega)$, $V_{h0}(\Omega) \subset H_0^1(\Omega)$. Given $G \subset \Omega$, $V_h(G)$ and $\mathcal{T}_h(G)$ are the restrictions of $V_h(\Omega)$ and $\mathcal{T}_h(\Omega)$ to G, respectively. Set

$$V_{h0}(G) = \{\, v \in V_{h0}(\Omega) \,:\, \operatorname{supp} v \subset \bar{G} \,\}. \tag{2.1}$$

For any $G \subset \Omega$ mentioned in this paper, we assume that it aligns with $\mathcal{T}_h(\Omega)$ when it is necessary.

For nonnegative integer k and $T \in \mathcal{T}_h(\Omega)$, let $P_k(T)$ denote the set of all polynomials with degree not greater than k. Let Π_T^1 be the linear interpolation operator with the function values at three vertices of T.

For $p \in P_T$, define $\partial_{ij,T}p \in P_1(T)$, $i,j \in \{1,2\}$, such that $\partial_{12,T}p = \partial_{21,T}p$ and for any $q \in P_1(T)$,

$$\begin{cases} \displaystyle\int_T q\partial_{11,T}p = \int_{\partial T} q\Pi_T^1\partial_1 p\nu_1 - \int_T \partial_1 q\,\partial_1 p, \\ \displaystyle\int_T q\partial_{22,T}p = \int_{\partial T} q\Pi_T^1\partial_2 p\nu_2 - \int_T \partial_2 q\,\partial_2 p, \\ \displaystyle 2\int_T q\partial_{12,T}p = \int_{\partial T} q\big(\Pi_T^1\partial_2 p\nu_1 + \Pi_T^1\partial_1 p\nu_2\big) - \int_T \big(\partial_2 q\,\partial_1 p + \partial_1 q\,\partial_2 p\big). \end{cases} \tag{2.2}$$