Journal of Computational Mathematics, Vol.26, No.2, 2008, 240–249.

THE RESTRICTIVELY PRECONDITIONED CONJUGATE GRADIENT METHODS ON NORMAL RESIDUAL FOR BLOCK TWO-BY-TWO LINEAR SYSTEMS*

Junfeng Yin and Zhongzhi Bai
LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China
Email: yinjf@lsec.cc.ac.cn, bzz@lsec.cc.ac.cn

Abstract

The *restrictively preconditioned conjugate gradient* (RPCG) method is further developed to solve large sparse system of linear equations of a block two-by-two structure. The basic idea of this new approach is that we apply the RPCG method to the normal-residual equation of the block two-by-two linear system and construct each required approximate matrix by making use of the incomplete orthogonal factorization of the involved matrix blocks. Numerical experiments show that the new method, called the *restrictively preconditioned conjugate gradient on normal residual* (RPCGNR), is more robust and effective than either the known RPCG method or the standard *conjugate gradient on normal residual* (CGNR) method when being used for solving the large sparse saddle point problems.

Mathematics subject classification: 65F10, 65W05.
Key words: Block two-by-two linear system, Saddle point problem, Restrictively preconditioned conjugate gradient method, Normal-residual equation, Incomplete orthogonal factorization.

1. Introduction

Consider an iterative solution of the *block two-by-two* (BTT) system of linear equations

$$Ax = b, \quad \text{where} \quad A = \begin{bmatrix} B & E \\ F & C \end{bmatrix} \in \mathbb{R}^{n\times n} \quad \text{and} \quad x, b \in \mathbb{R}^n, \tag{1.1}$$

with $B \in \mathbb{R}^{m\times m}$ and $C \in \mathbb{R}^{l\times l}$ being large, sparse, square and nonsymmetric matrices, $E \in \mathbb{R}^{m\times l}$ and $F \in \mathbb{R}^{l\times m}$ being sparse matrices, with $m \geq l$, such that $A \in \mathbb{R}^{n\times n}$ is nonsingular, where $n = m + l$. The BTT linear system (1.1) frequently arises in many areas of scientific computing and engineering applications such as the constrained least-squares problems, the Navier-Stokes equations in fluid computations, and the Maxwell equations in computational electromagnetics; see [1, 7, 2] for more details. Obviously, the saddle point problems form a subset of the BTT linear systems.

When the matrix blocks B is symmetric positive definite, C is symmetric positive semidefinite (e.g., $C = 0$) and $F = \pm E^T$, Bai and Li [4] recently proposed the *restrictively preconditioned conjugate gradient* (RPCG) methods for these special forms of the BTT linear system (1.1) and studied their convergence properties. Numerical results showed that this class of methods are robust and effective solvers for iteratively computing the solutions of the symmetric positive definite and the Hamiltonian systems of linear equations. We remark that the RPCG methods

* Received January 20, 2006 / Revised version received April 2, 2007 / Accepted September 19, 2007 /

were also developed to general nonsingular and nonsymmetric linear systems, resulting in a rather general framework of iterative methods, which not only covers many standard Krylov subspace methods such as the conjugate gradient [8, 11], the conjugate residual [8, 9], the conjugate gradient on normal residual (CGNR) [8, 9, 13] and the conjugate gradient on normal equation (CGNE) [8, 9, 13], as well as their preconditioned variants, but also yields many new ones. For details, we refer to [4] and references therein. Latter, Bai and Wang [5] further developed the RPCG method and obtained an inexact variant for the symmetric positive definite case of the BTT linear system (1.1), in which both B and C are symmetric positive definite and $F = E^T$.

In this paper, we use the inexact RPCG method presented in [5] to solve the BTT linear system (1.1). This new approach first forms the normal-residual equation

$$\mathcal{A}x \equiv A^T A x = A^T b \equiv \mathbf{b}, \tag{1.2}$$

where

$$\mathcal{A} = \begin{bmatrix} \mathcal{B} & \mathcal{E} \\ \mathcal{E}^T & \mathcal{C} \end{bmatrix} \equiv \begin{bmatrix} B^T B + F^T F & B^T E + F^T C \\ E^T B + C^T F & C^T C + E^T E \end{bmatrix} \tag{1.3}$$

is symmetric positive definite, with

$$\mathcal{B} = B^T B + F^T F, \quad \mathcal{C} = C^T C + E^T E \quad \text{and} \quad \mathcal{E} = B^T E + F^T C, \tag{1.4}$$

and then straightforwardly apply the inexact RPCG method developed in [5] to (1.2)-(1.4), as now $\mathcal{B} \in \mathbb{R}^{m\times m}$ and $\mathcal{C} \in \mathbb{R}^{l\times l}$ are symmetric positive definite and $\mathcal{E} \in \mathbb{R}^{m\times l}$ is of full column rank.

For the saddle point problems of the form

$$Ax = b, \quad \text{where} \quad A = \begin{bmatrix} B & E \\ E^T & 0 \end{bmatrix} \in \mathbb{R}^{n\times n} \quad \text{and} \quad x, b \in \mathbb{R}^n, \tag{1.5}$$

with $B \in \mathbb{R}^{m\times m}$ being a positive definite matrix, $E \in \mathbb{R}^{m\times l}$ being a matrix of full column rank and $m \geq l$, we give a practical way for constructing approximations to the submatrices $\mathcal{B}$ and $\mathcal{S} = \mathcal{C} - \mathcal{E}^T \mathcal{B}^{-1} \mathcal{E}$ by utilizing the incomplete orthogonal factorization technique in [3]; see also [12, 6]. The resulting method, called *RPCG on normal residual* (RPCGNR), is algorithmically described in detail. Numerical examples are used to show that the RPCGNR method outperforms the *preconditioned CGNR* (PCGNR) method [8] when they are employed to solve the large sparse saddle point problem (1.5). Moreover, RPCGNR also shows better numerical behaviour than both RPCG [4] and PCGNR when the matrix block B is symmetric positive definite and when RPCG is directly applied to the saddle point problem (1.5).

The organization of this paper is as follows. After establishing the RPCGNR method for solving the BTT linear system (1.1) and its special case (1.5) in Section 2, we present practical choices for approximating matrix blocks involved in the saddle point problem (1.5) in Section 3. In Section 4, numerical results are used to show the feasibility and effectiveness of our new method. Finally, in Section 5, we end this paper with a brief conclusion.

2. The RPCGNR Method

To establish the RPCGNR method for solving the BTT linear system (1.1), according to [5] we first decompose the block two-by-two matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ in (1.3) as $\mathcal{A} = \mathcal{PHQ}$, where