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# AN EXPLICIT MULTI-CONSERVATION FINITE-DIFFERENCE SCHEME FOR SHALLOW-WATER-WAVE EQUATION*

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**Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday**

**Abstract**

An explicit multi-conservation finite-difference scheme for solving the spherical shallow-water-wave equation set of barotropic atmosphere has been proposed. The numerical scheme is based on a special semi-discrete form of the equations that conserves four basic physical integrals including the total energy, total mass, total potential vorticity and total enstrophy. Numerical tests show that the new scheme performs closely like but is much more time-saving than the implicit multi-conservation scheme.

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*Key words:* Explicit finite difference scheme, Multi-conservation, Shallow-water-wave, Physical integral.

## 1. Introduction

The spherical shallow-water-wave equation set of barotropic atmosphere, a representative atmospheric equation set, conserves five basic physical integrals including the total energy, total mass, total vorticity, total enstrophy and total angular momentum. These constant integrals imply important physical characteristic and mathematical significance of atmospheric motions [1,2]. To conserve these integrals as many as possible in a discrete scheme of the equation set is very necessary, which is one of the essential criterions to evaluate the scheme. For this reason, many efforts have been made on designing multi-conservation schemes for atmospheric equations [3–6]. The available multi-conservation schemes, however, are implicit and time-consuming due to a number of iterations for getting their solutions. Whether and how can an explicit multi-conservation scheme be constructed? This is an interesting question. A significant attempt to design an explicit multi-conservation finite-difference scheme is made in this paper, based on a special semi-discrete form of the spherical shallow-water-wave equation set of barotropic atmosphere that was applied to construct an implicit scheme with 4 conservation properties by Wang and Ji [6].

## 2. Equations and Conservations

The shallow-water-wave equation set of barotropic atmosphere in spherical coordinate system is originally formulated as

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$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{1}{a\cos\theta}\left[\dfrac{\partial\varphi}{\partial\lambda} + u\dfrac{\partial u}{\partial\lambda} + v^*\dfrac{\partial u}{\partial\theta}\right] + f^*v, \\ \dfrac{\partial v}{\partial t} = -\dfrac{1}{a\cos\theta}\left[\cos\theta\dfrac{\partial\varphi}{\partial\theta} + u\dfrac{\partial v}{\partial\lambda} + v^*\dfrac{\partial v}{\partial\theta}\right] - f^*u\ , \\ \dfrac{\partial\varphi}{\partial t} = -\dfrac{1}{a\cos\theta}\left[\dfrac{\partial}{\partial\lambda}(u\varphi) + \dfrac{\partial}{\partial\theta}(v^*\varphi)\right]\ , \end{cases} \tag{2.1}$$

where $\theta$, $\lambda$ are the latitude and longitude respectively; $a$ denotes the radius of earth, $u$,$v$ and $\varphi$ represent the zonal wind, meridional wind and geopotential height; $v^* = v\cos\theta$; $f^* = 2\omega_0\sin\theta + ua^{-1}\tan\theta$, $\omega_0$ is the angular velocity of the earth. It can be expressed into a concise form:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{a\cos\theta}\dfrac{\partial E}{\partial\lambda} - \eta v = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{1}{a}\dfrac{\partial E}{\partial\theta} + \eta u = 0, \\ \dfrac{\partial\varphi}{\partial t} + \dfrac{1}{a\cos\theta}\left[\dfrac{\partial}{\partial\lambda}(u\varphi) + \dfrac{\partial}{\partial\theta}(v^*\varphi)\right] = 0, \end{cases} \tag{2.2}$$

where

$$\begin{cases} E = \dfrac{1}{2}\left(u^2 + v^2\right) + \varphi, \\ \eta = \dfrac{1}{a\cos\theta}\left[\dfrac{\partial v}{\partial\lambda} - \dfrac{\partial\Omega}{\partial\theta}\right] = q + 2\omega\sin\theta, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \Omega = u\cos\theta + a\omega\cos^2\theta, \\ q = \dfrac{1}{a\cos\theta}\left[\dfrac{\partial v}{\partial\lambda} - \dfrac{\partial}{\partial\theta}(u\cos\theta)\right]. \end{cases} \tag{2.4}$$

It is easy to prove that the equation set has five basic constant integrals [6]:

$$\begin{cases} \dfrac{\partial}{\partial t}\iint\limits_D \left(e + \dfrac{1}{2}\varphi\right)\varphi ds = 0, & \dfrac{\partial}{\partial t}\iint\limits_D \varphi ds = 0, \\ \dfrac{\partial}{\partial t}\iint\limits_D \xi\varphi ds = 0, & \dfrac{\partial}{\partial t}\iint\limits_D \xi^2\varphi ds = 0, \\ \dfrac{\partial}{\partial t}\iint\limits_D \Omega\varphi ds = 0, & \end{cases} \tag{2.5}$$

where the area unit $ds$ is defined as: $ds = a^2\cos\theta d\lambda d\theta$, D is the integration region (here it is the whole spherical surface), $e$ and $\xi$ are respectively the kinetic energy and potential vorticity, which are defined as follows:

$$e = \frac{1}{2}\left(u^2 + v^2\right), \quad \xi = \eta/\varphi. \tag{2.6}$$

## 3. Semi-discrete Equation Set

After discretizing the spatial differential terms of Eq. (2.2), a semi-discrete spherical shallow-water-wave equation set of barotropic atmosphere on an Arakawa A-grid system has been