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A NOTE ON RICHARDSON EXTRAPOLATION OF GALERKIN METHODS FOR EIGENVALUE PROBLEMS OF FREDHOLM INTEGRAL EQUATIONS*

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Abstract

In this paper, we introduce a new extrapolation formula by combining Richardson extrapolation and Sloan iteration algorithms. Using this extrapolation formula, we obtain some asymptotic expansions of the Galerkin finite element method for semi-simple eigenvalue problems of Fredholm integral equations of the second kind and improve the accuracy of the numerical approximations of the corresponding eigenvalues. Some numerical experiments are carried out to demonstrate the effectiveness of our new method and to confirm our theoretical results.

Mathematics subject classification: 65N25, 65N30, 65R20.
Key words: Fredholm integral equations, Semi-simple eigenvalues, Asymptotic expansion, Galerkin method, Richardson extrapolation, Sloan iteration.

1. Introduction

In this paper, we consider the eigenvalue problem of the Fredholm integral equation of the second kind: Find an eigen-pair $(\lambda, u) \in R \times L^2(\Omega)$, such that

$$\int_\Omega k(t,s)u(s)ds = \lambda u(t), \quad t,s \in \Omega \subseteq R^n, \tag{1.1}$$

where $k(s,t)$ is a given smooth function in $D := \Omega \times \Omega$ satisfying $k(t,s) = k(s,t)$. Let T be an integral operator defined by:

$$(Tu)(t) = \int_\Omega k(t,s)u(s)ds, \quad t,s \in \Omega.$$

The corresponding operator form of (1.1) is

$$(Tu)(t) = \lambda u(t), \quad t \in \Omega.$$

Then the integral operator T is self-adjoint and compact. Thus the eigenvalues λ of T are semi-simple, i.e., the algebraic multiplicity of λ equals the geometric multiplicity of λ.

Let the algebraic multiplicity of the semi-simple eigenvalue λ be r. Then there are r numerical eigenvalues approximating λ. The authors of [9] established asymptotic expansions for

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arithmetic mean of r eigenvalues (approximating λ) in the Galerkin method. Lin, Sloan and Xie [12] proved similar results for the solution of Fredholm equations of the second kind. Mclean [14] also discussed asymptotic error expansions for the solution of Fredholm equations of the second kind.

There have been many attempts in improving the accuracy of numerical solutions. The most popular methods include the Sloan iteration method (iteration post-processing method), interpolation post-processing method and the Richardson extrapolation, see, e.g., [2, 5, 9, 11, 13, 15].

Suppose that the Galerkin eigen-pair (λ_h, u_h) of degree $m-1$ approximates (λ, u). In a recent work [17], the authors derived an asymptotic expansion of the eigenvalue approximation error for Problem (1.1) by means of iterated Galerkin finite element methods in certain piecewise polynomial spaces:

$$\begin{aligned}\lambda - \lambda_h &= (u, T(I-P_h)u) + \mathcal{O}(h^{3m}) \\ &= \beta_u h^{2m} + \mathcal{O}(h^{2m+2}) + \mathcal{O}(h^{3m}),\end{aligned}$$

where β_u depends only on the eigenfunction u. Replacing h with $h/2$ for the above equation and extrapolating between λ_h and $\lambda_{h/2}$, the authors obtained a higher order approximation for a simple eigenvalue λ:

$$\begin{aligned}&\frac{2^{2m}\lambda_{h/2} - \lambda_h}{2^{2m}-1} = \lambda + \mathcal{O}(h^{2m+2}),\ m \geq 2; \\ &\frac{4\lambda_{h/2} - \lambda_h}{3} = \lambda + \mathcal{O}(h^3),\ m = 1.\end{aligned}$$

In order to use the method in [17], it is crucial that the eigenvalue λ is simple, so that the u approximated by u_h and the u approximated by $u_{h/2}$ are the same. However, this is not the case for semi-simple eigenvalues, which occur when we solve boundary integral equations and high dimensional integral equations. Therefore, there is a need to develop a new method for higher order approximations of semi-simple eigenvalues.

In this note we propose the following procedure.

1) Calculate eigen-pair (λ_h, u_h) by the Galerkin finite element method.
2) Apply the Sloan iteration to u_h to obtain $u_h^s = \lambda_h^{-1} T u_h$ and the normalized $\widetilde{u_h^s} = u_h^s / \|u_h^s\|$.
3) Project $\widetilde{u_h^s}$ onto the bi-sectioned mesh and obtain $\overline{u}^s = P_{h/2}\widetilde{u_h^s}$.
4) Calculate another eigenvalue approximation $\lambda^s = (T\overline{u}^s, \widetilde{u_h^s})$.
5) Extrapolate between λ_h and λ^s to achieve higher-order eigenvalue approximation.

The main advantage of our new method is twofold. First, it is applicable to both simple and semi-simple eigenvalues as well as higher-dimensional cases. Second, comparing with the traditional Richardson extrapolation between λ_h and $\lambda_{h/2}$, extrapolation between λ_h and λ^s is much cheaper. Note that the cost for $\lambda_{h/2}$ is $8n^3$ times of the cost for λ_h in case of a typical n-dimensional eigenvalue problem.

Here is the outline of the remaining sections. In Section 2, we state our main results. To prove these results, we list some relevant lemmas in Section 3. Section 4 provides proofs of the main theorems. Finally, numerical results are presented in Section 5.