Journal of Computational Mathematics, Vol. 25, No. 3, 2007, 350–367.

THE USE OF PLANE WAVES TO APPROXIMATE WAVE PROPAGATION IN ANISOTROPIC MEDIA*

Tomi Huttunen
(*Department of Physics, University of Kuopio, P.O. Box 1627, FIN-70211 Kuopio, Finland*
Email: tomi.huttunen@uku.fi)

Peter Monk
(*Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA*
Email: monk@math.udel.edu)

Abstract

In this paper we extend the standard Ultra Weak Variational Formulation (UWVF) of Maxwell's equations in an isotropic medium to the case of an anisotropic medium. We verify that the underlying theoretical framework carries over to anisotropic media (however error estimates are not yet available) and completely describe the new scheme. We then consider TM mode scattering, show how this results in a Helmholtz equation in two dimensions with an anisotropic coefficient and demonstrate how to formulate the UWVF for it. In one special case, convergence can be proved. We then show some numerical results that suggest that the UWVF can successfully simulate wave propagation in anisotropic media.

Mathematics subject classification: 65N30, 65N12
Key words: Ultra weak, Maxwell, Plane wave, Anisotropic medium.

1. Introduction

Electromagnetic wave propagation in anisotropic media arises in several applications including ground penetrating radar [3], microwave interaction with wood [13] and biological materials [20]. This paper is devoted to developing a method for approximating the electromagnetic field propagating in anisotropic media, with particular attention to microwave interactions with anisotropic (e.g. wooden) scatterers. This implies that the wavelength of the radiation is neither very large nor very small compared to features of scatterers located in the medium.

We shall develop a Discontinuous Galerkin (DG) method for the anisotropic Maxwell system with the novelty that local solutions of the anisotropic Maxwell system on each element are used as basis functions. This requires us to impose the restriction that the matrix electromagnetic parameters ϵ (permittivity) and μ (permeability) must be piecewise constant on each element in the mesh. More precisely, the method we shall develop is an extension of the Ultra Weak Variational Formulation (UWVF) of Cessenat and Després [4–6] to anisotropic media. The basic UWVF has proved to be a convenient method for approximating electromagnetic scattering in isotropic media. For example, in [17] we detail the connection of the UWVF to standard DG methods, give several extensions to the basic UWVF and provide validation results using standard electromagnetic scattering benchmark problems.

It is, of course, possible to handle anisotropic media in the classical finite element method for Maxwell's equations based on Nédélec's [26] edge elements. We have found the UWVF to

* Received November 6, 2006; Final revised February 1, 2007.

be competitive with the edge finite element method, and in [17] we show that the UWVF can be more memory efficient then edge elements and is also easily parallelized for electromagnetic applications. This motivates our extension of the UWVF to anisotropic media.

The UWVF is by no means the only technique that can be used to approximated wave propagation by using local solutions of the Maxwell system as basis functions. The UWVF resembles the basic Trefftz type finite element technique that has been applied to Maxwell's equations in [27, 28] although the variational statement is different. At least for the Helmholtz equation, other methods include least squares techniques [21, 25, 30] and enriched finite element methods [31, 32]. In a different direction, the Partition of Unity Finite Element Method (PUFEM) constructs a conforming approximation space as a product of partition of unity functions (usually finite element hat functions) and plane waves [1, 22, 23]. As yet there is probably not enough experience with the various methods to declare one preferable to another (for interesting comparison results for the Helmholtz equation in 2D see [12, 14]).

The plan of this paper is as follows. In the remainder of the introduction we shall describe in more detail the problem we shall study. Then in Section 2 we show that the fundamental mathematical result behind the UWVF for Maxwell's equations still holds for anisotropic media. This implies that the basic UWVF can be extended to anisotropic media, and we give details. In Section 3 we examine the choice of basis functions. As usual we employ a basis of plane waves on each element [4]. Plane wave propagation in anisotropic media is a classical topic in textbooks on electromagnetism (see e.g. [7, 19]). We shall summarize some of the relevant results, derive some consequences for the UWVF and then show how to use the plane waves to discretize the anisotropic UWVF. Almost all theoretical questions related to the 3D UWVF approach to anisotropic media are still open: in particular, the relevant approximation properties of sums of anisotropic plane waves are not known.

In Section 4 we discuss the case of electromagnetic wave propagation in an orthotropic medium. This reduces to solving the Helmholtz equation in two dimensions with an anisotropic "diffusion" coefficient. This problem can also be approximated by an extension of the UWVF in [4] to orthotropic media (or by a restriction of the above mentioned 3D UWVF to the orthotropic case). In contrast to the 3D case, the known convergence theory for the 2D Helmholtz UWVF can be extended to the anisotropic case in one case. We summarize the main steps and results.

In Section 5 we present some preliminary numerical results for an orthotropic medium. These results show that the UWVF is a promising method for dealing with anisotropic media. One interesting result is that since we can view the Perfectly Matched Layer (PML) [2, 24] as a special (non-physical) anisotropic medium, it can be implemented in a general anisotropic code without further modification. Our results show that this method of implementing the PML works as well as our previous implementation based on special plane waves [16, 17].

The model problem we shall investigate is to approximate the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ (appropriately scaled [24]) that satisfy the Maxwell system

$$\left.\begin{array}{r}-ik\epsilon_r\boldsymbol{E}-\nabla\times\boldsymbol{H}=0\\ -ik\mu_r\boldsymbol{H}+\nabla\times\boldsymbol{E}=0\end{array}\right\}\quad\text{in }\Omega,\tag{1.1}$$

where Ω is a bounded polyhedral domain. Here k is the wave-number of the radiation that is related to the temporal frequency $\omega > 0$, the permittivity of free space ϵ_0 and the permeability of free space μ_0 by $k = \omega\sqrt{\epsilon_0\mu_0}$. The relative permittivity ϵ_r is assumed to be a complex valued