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MULTIVARIATE FOURIER TRANSFORM METHODS OVER SIMPLEX AND SUPER-SIMPLEX DOMAINS*1)

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

In this paper we propose the well-known Fourier method on some non-tensor product domains in R^d, including simplex and so-called super-simplex which consists of $(d+1)!$ simplices. As two examples, in 2-D and 3-D case a super-simplex is shown as a parallel hexagon and a parallel quadrilateral dodecahedron, respectively. We have extended most of concepts and results of the traditional Fourier methods on multivariate cases, such as Fourier basis system, Fourier series, discrete Fourier transform (DFT) and its fast algorithm (FFT) on the super-simplex, as well as generalized sine and cosine transforms (DST, DCT) and related fast algorithms over a simplex. The relationship between the basic orthogonal system and eigen-functions of a Laplacian-like operator over these domains is explored.

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1. Introduction

It is well known that the univariate Fourier transform and its tensor product form in high dimension have been played a key role in applied mathematics and scientific computations in many fields. Unfortunately, in high dimension the tensor product approach has to be limited to domains which can transformed into a box. With theoretical interests and application drives for developing more efficient grid generation and fast solver for numerical PDE, there have been increasing interests in how to extend Fourier methods into irregular domains. Since simplex is one of the simplest non-box domains, it seems more attention has being paid to the multivariate orthogonal polynomials on simplex [1, 3, 4, 5, 10, 11, 12, 13]. In this paper we also study so-called generalized sine and cosine transform and related fast transforms on a simplex.

As is well-known a simplex is a natural extension of an interval $[0, 1]$ in $1 - D$. However, besides simplex there is another kind of domain partition which is really important for high dimension case. In 2-D and 3-D case, they are parallel hexagon and parallel dodecahedrons which can be taken as a natural extension of the symmetry interval $[-1, -1]$ instead of a general interval $[0, 1]$. In application point of views, this kind of domain partitions exist in nature almost everywhere. As a well-known example, the shape of honeycombs is a hexagon instead of quadrilaterals. There are many parallel dodecahedron shapes in natural crystals.

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Mathematically, it is worth to note that besides a box partition with d-directions, such a super-simplex partitions $d+1$-directions can also form a tiling in $\mathbb{R}^d$. However, a pure simplex partition can not form a tiling if only shifting operators are allowed.

The most important issue of the generalization of Fourier transform for non-box domains is the construction of the orthogonal bases. There is an intrinsic relationship between Fourier basis and the Laplacian-like operator. As is well known, the eigen-decompositions of the Laplacian-like operator usually result in the orthogonal exponential bases on box domains, instead of orthogonal polynomials. To construct orthogonal bases in our case, in this paper we adopt eigen-decompositions of the differential operators with periodic boundary conditions over the super-simplex domain or with zero Dirichlet and zero Neumann boundary conditions. The idea and algorithms may be useful for spectral methods and preconditioning algorithms for numerical PDE solvers for non separable problems and over more wide non tensor-product partitions.

The remainder of this paper is organized as follows. In §2, after introducing some homogeneous coordinates, we present a tiling of the $d+1$-direction partition in $\mathbb{R}^d$. In §3, we construct a basis and investigate its some properties, such as periodicity, orthogonality and completeness. Then we study the related Fourier series and its convergence. We put our focus on exploring the intrinsic relationship between the above basic system and eigenfunctions of a Laplacian-Like PDE operator. On the analogy of univariate case, so-called generalized sine (HSin) and cosine functions (HCos) are defined in the whole space, especially on a simplex in §4. Finally we investigate the generalized Fourier transform (HFT) and propose a fast discrete Fourier transform (HFFT) on §5.

2. Notations and Definitions

Suppose vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d$ are linear independent in $\mathbb{R}^d$ and G is the Gram matrix

$$G(\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d) = \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{e}_2 & ... & \mathbf{e}_1 \cdot \mathbf{e}_d \\ \mathbf{e}_2 \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{e}_2 & ... & \mathbf{e}_2 \cdot \mathbf{e}_d \\ ... & ... & ... & ... \\ \mathbf{e}_d \cdot \mathbf{e}_1 & \mathbf{e}_d \cdot \mathbf{e}_2 & ... & \mathbf{e}_d \cdot \mathbf{e}_d \end{pmatrix} = [\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d]^T [\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d] \tag{2.1}$$

where the notation $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1' \mathbf{e}_2$ denotes the inner product of $\mathbf{e}_1$ and $\mathbf{e}_2$ in $\mathbb{R}^d$.

We define a set of vectors as

$$[\mathbf{n}_1, \mathbf{n}_2, ..., \mathbf{n}_d] = [\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d][G(\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d)]^{-1}. \tag{2.2}$$

Then two vector sets $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d\}$ and $\{\mathbf{n}_1, \mathbf{n}_2, ..., \mathbf{n}_d\}$ are bi-orthogonal in the sense

$$[\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d]^T [\mathbf{n}_1, \mathbf{n}_2, ..., \mathbf{n}_d] = [\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d]^T [\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d][G(\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d)]^{-1} = I.$$

Moreover, we define the $(d+1)$-th direction vector

$$\mathbf{n}_{d+1} = -\sum_{\mu=1}^{d} \mathbf{n}_\mu. \tag{2.3}$$

Hence there are totally $2\binom{d+1}{d-1} = d(d+1)$ hyperplanes via the $d+1$ directions $\mathbf{n}_1, ..., \mathbf{n}_d, \mathbf{n}_{d+1}$. Now we lift the space $\mathbb{R}^d$ to $\mathbb{R}^{\binom{d+1}{2}}$. Set

$$\mathbf{n}_{\nu,\nu} = \mathbf{n}_\nu \quad (\nu = 1, ...d), \qquad \mathbf{n}_{\mu,\nu} = \mathbf{n}_\nu - \mathbf{n}_\mu \quad (1 \le \mu < \nu \le d). \tag{2.4}$$

For each point $P \in \mathbb{R}^d$, we introduce its $\binom{d+1}{2}$ affine coordinates by taking its corresponding vector projections in the space $\mathbb{R}^d$ as

$$t_{\nu,\nu} = t_\nu = \mathbf{P} \cdot \mathbf{n}_\nu \quad (\nu = 1, ...d), \qquad t_{\mu,\nu} = \mathbf{P} \cdot \mathbf{n}_{\mu,\nu} \quad (1 \le \mu < \nu \le d) \tag{2.5}$$

with the similar relations to (2.4) $t_{\mu,\nu} = t_\nu - t_\mu \quad (1 \le \mu < \nu \le d)$.