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CONVERGENCE ANALYSIS FOR A NONCONFORMING MEMBRANE ELEMENT ON ANISOTROPIC MESHES *1)

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Abstract

Regular assumption of finite element meshes is a basic condition of most analysis of finite element approximations both for conventional conforming elements and nonconforming elements. The aim of this paper is to present a novel approach of dealing with the approximation of a four-degree nonconforming finite element for the second order elliptic problems on the anisotropic meshes. The optimal error estimates of energy norm and L^2-norm without the regular assumption or quasi-uniform assumption are obtained based on some new special features of this element discovered herein. Numerical results are given to demonstrate validity of our theoretical analysis.

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Key words: Anisotropic mesh, Nonconforming finite element, Optimal estimate.

1. Introduction

It is well known that Carey's element [1] is a very famous four-degree triangle nonconforming membrane element. Numerous studies have been advocated to its convergence analysis (see [2],[3] and [4] for details). However, one of the drawbacks of the analysis of convergence of above studies is that the regular assumption of the finite element meshes should be satisfied , i.e. there exists a constant $C > 0$, such that for all element K, $h_K/\rho_K \leq C$, where h_K and ρ_K are the diameter of K and the biggest circle contained in K respectively. Therefore, this restriction limits the applications of many elements of practical problems. In practice, the solution of elliptic boundary problem may have anisotropic behavior in parts of the domain, i.e. it varies significantly only in certain direction. In such cases it is an obvious idea to reflect this anisotropy in the discretization by using anisotropic meshes with small mesh size in the direction of rapid variation of solution and a larger mesh size in the perpendicular direction.

In recent years, some researchers have been interested in the study of theoretical analysis and computations without the above regular assumption, i.e. anisotropic behavior, and paid more attention to the interpolation error estimate of conforming Lagrange type elements [5,6,10] and nonconforming C-R type element [7] with narrow edges or having anisotropic properties. In these cases, the key problem is that the usual Sobolev theories (for example, Hilbert-Bramble Lemma) can not be used directly. An example is given in [7].

In this paper, we focus on the study of convergence analysis of Carey's element with the narrow edges or anisotropic properties. The optimal error estimates are obtained by using Lagrange interpolation results for conforming elements and some new properties discovered

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herein. The results and the approach of this paper are also valid for some other elements, such as the Wilson element [4], the arbitrary quadrilateral quasi-Wilson element [8,9] and five-node element [12] and so on . At the same time, we also present some computational results which demonstrate the validity and coincidence of our theoretical analysis very well.

2. Carey's Element and Some Lemmas

Let K be a triangle with vertices $p_i = (x_i, y_i), 1 \leq i \leq 3$, and λ_i be the area coordinate corresponding to the vertices p_i, $\ell_i = \overrightarrow{p_{i+1}p_{i+2}}$, $(i = 1, 2, 3, mod(3))$ be the three sides. Let S denote the area of the triangle K and set the following remarks

$$\begin{cases} \xi_1 = x_2 - x_3, & \xi_2 = x_3 - x_1, & \xi_3 = x_1 - x_2, \\ \eta_1 = y_2 - y_3, & \eta_2 = y_3 - y_1, & \eta_3 = y_1 - y_2, \\ \xi_i^2 + \eta_i^2 = \ell_i^2, & \ell^2 = \ell_1^2 + \ell_2^2 + \ell_3^2. \end{cases}$$

Then, the shape function on element K may be found by

$$u = \sum_{i=1}^{3} u_i \lambda_i + t(u)\varphi, \tag{1}$$

where

$$\varphi = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \tag{2}$$

and u_i denotes the functional value of u at the vertex p_i (i=1,2,3) of K respectively, and the parameter $t(u)$ is taken as

$$t(u) = \frac{-4S}{\ell^2} \int_K \Delta u dx dy. \tag{3}$$

Obviously, this element is a nonconforming membrane element and it is continuous at each vertex of the element K. Let

$$\overline{u} = u_1\lambda_1 + u_2\lambda_2 + u_3\lambda_3, \quad u^1 = t(u)\varphi. \tag{4}$$

Then, $u = \bar{u} + u^1$, i.e. $\bar{u}$ and u^1 are the conforming part and nonconforming part of u respectively.

Let Ω be the polygonal domain, J_h be a family of decomposition of Ω with $\overline{\Omega} = \bigcup_{K \in J_h} \overline{K}$, and $diam(K) \leq h$,$\forall$ $K \in J_h$. For a given element $K \in J_h$, let $\overrightarrow{p_1p_2}$ be the longest edge of K. Then we denote $h_1 = h_{1,K} = meas(\overrightarrow{p_1p_2})$ its length and by $h_2 = h_{2,K} = \frac{2S}{h_{1,K}}$ the thickness of K perpendicularly to $\overrightarrow{p_1p_2}$. We assume that the element satisfies the maximum angle condition and a coordinate system condition [11], but it is not necessary to satisfy the regular assumption or quasi-uniform assumptions on meshes [4]. Let F_K be an affine mapping from $\hat{K}$ to K

$$F_K : \quad \begin{cases} x = \sum_{j=1}^{3} x(p_j)\lambda_j, \\ y = \sum_{j=1}^{3} y(p_j)\lambda_j. \end{cases}$$

Let V_h be the associated Carey's finite element space.

$$V_h = \{ \ v : v|_K = \hat{v} \circ F_K^{-1}, \ \hat{v} \in P_K, \ v(a) = 0, \ \forall \ node \ a \in \partial\Omega \ \},$$

where $P_K = span\{\lambda_1, \lambda_2, \lambda_3, \varphi\}$ is the shape function space.