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AD GALERKIN ANALYSIS FOR NONLINEAR PSEUDO-HYPERBOLIC EQUATIONS*1)

Xia Cui
(*Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009-26, Beijing 100088, China*)

Abstract

AD (Alternating direction) Galerkin schemes for d-dimensional nonlinear pseudo-hyperbolic equations are studied. By using patch approximation technique, AD procedure is realized, and calculation work is simplified. By using Galerkin approach, highly computational accuracy is kept. By using various priori estimate techniques for differential equations, difficulty coming from non-linearity is treated, and optimal H^1 and L^2 convergence properties are demonstrated. Moreover, although all the existed AD Galerkin schemes using patch approximation are limited to have only one order accuracy in time increment, yet the schemes formulated in this paper have second order accuracy in it. This implies an essential advancement in AD Galerkin analysis.

Key words: nonlinear, pseudo-hyperbolic equation, alternating direction, numerical analysis

1. Introduction

Consider the nonlinear pseudo-hyperbolic equation with memory term given by

$$\begin{aligned} q(u)u_{tt} &= \nabla\cdot(a(u)\nabla u_t + b(u)\nabla u \\ &+ \textstyle\int_0^t c(u(\tau))\nabla u(\tau)d\tau) + p(u)\nabla u_t + r(u)\nabla u + f(u), && x\in\Omega, t\in J. \\ u(x,t) &= 0, && x\in\partial\Omega, t\in J. \\ u(x,0) &= u_0(x), u_t(x,0) = u_{t0}(x), && x\in\Omega. \end{aligned} \tag{1.1}$$

where $\Omega\subset R^d$ ($d\geq 2$ is the dimension of the space) is an open bounded domain with piecewise smooth boundary $\partial\Omega$. $x=(x_1,\cdots,x_d)$. $J=[0,T]$. $\phi(u)=\phi(x,t,u)$ for $\phi=q,a,b,p,r,f$, $c(u(\tau))=c(t,\tau,x,u(x,\tau))$, and $u_0(x)$, $u_{t0}(x)$ are known functions.

We assume that:

1) there exist positive constants q^*,q_*,a^* and a_* such that $q^*\geq q(x,t,\psi)\geq q_*, a^*\geq a(x,t,\psi)\geq a_*$, for all $x\in\Omega, t\in J, \psi\in R$.

2) the function q is Lipschitz continuous with respect to t and u.

3) the functions b,c are bounded, a,b,c and the derivatives c_u, c_τ, $c_{\tau\tau}$ are Lipschitz continuous with respect to t, the derivatives a_u, b_u are Lipschitz continuous with respect to t and u.

4) the functions p,r are bounded, p,r and f are Lipschitz continuous with respect to u.

Equation (1.1) is also called pseudo-hyperbolic integro-differential equation, it is widely used in the fields of visco-elasticity, nuclear physics and biological mechanics. There is some work on its qualitative analysis and numerical solution [3,4,11]. When the memory term $c(u(\tau))\equiv 0$, (1.1) is called pseudo-hyperbolic equation in a usual meaning, which often appears in visco-elasticity

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theory, for example, in the propagation of sound in viscous media and other phenomena of similar nature [1]. There are also some numerical methods for it [9]. But the existed numerical approaches for these two equations are limited to Galerkin schemes, which have highly accuracy, but need fairly complicated calculation. In this paper, we first consider their AD (alternating direction) Galerkin solutions. AD Galerkin method was first propounded by Douglas and Dupont [6,7], and was verified very efficient in numerical approach of parabolic and hyperbolic equations, it can keep highly accuracy of Galerkin method, and can solve large multi-dimensional problems as a series of smaller one-dimensional problems by AD technique, hence can simplify computational work. Here, we will use patch approximation [7] to treat $q(u)$, to realize AD procedure, and use various priori estimate techniques for differential equations to treat difficulty coming from non-linearity, and to obtain optimal H^1 and L^2 convergence of our schemes. Something deserving of mention is that all the existed AD Galerkin schemes using patch approximation are limited to have only one order accuracy in time increment, while the schemes established here have second order accuracy in it. This means an essential improvement in AD Galerkin analysis.

Since pseudo-hyperbolic equation can be regarded as a special case of pseudo-hyperbolic integro-differential equation, we may just study the numerical analysis for the latter, and let the approximation terms derived from $c(u(\tau))$ equal zero to obtain corresponding results for the former. Besides, before studying AD Galerkin scheme, we consider a Galerkin analysis first for convenience.

An outline of the paper is as follows. In Section 2, a Galerkin scheme and its convergence analysis are described. In Section 3, an AD Galerkin procedure and its analysis are given. In Section 4, start-up procedures for the preceding schemes are discussed and generalization is made.

In this paper, the letter K will be a generic constant, and may be different each time it is used. ϵ will be an arbitrarily small constant. Let $(\phi,\psi)=\int_\Omega \phi\psi dx$, and let the norms in the Banach space follow those in [7] and [8].

Divide $[0,T]$ into L small intervals with equal step length $\Delta t=\frac{T}{L}$, denote $t_l=l\Delta t$, $t_{l+\frac{1}{2}}=(l+\frac{1}{2})\Delta t$, $\phi^l=\phi(t_l)$, $\phi^{l+\frac{1}{2}}=\frac{1}{2}(\phi^{l+1}+\phi^l)$, $E\phi^l=2\phi^{l-1}-\phi^{l-2}$, $d_t\phi^l=\frac{1}{\Delta t}(\phi^{l+1}-\phi^l)$, $\partial_t\phi^l=\frac{1}{2\Delta t}(\phi^{l+1}-\phi^{l-1})$, and $\partial_{tt}\phi^l=\frac{1}{(\Delta t)^2}(\phi^{l+1}-2\phi^l+\phi^{l-1})$.

2. Analysis for Galerkin scheme

The weak form of (1.1) can be writen as

$$\begin{aligned} &w=u_t,\\ &(q(u)w_t,v)+(a(u)\nabla w+b(u)\nabla u+\int_0^t c(u(\tau))\nabla u(\tau)d\tau,\nabla v)\\ &-(p(u)\nabla w+r(u)\nabla u,v)=(f(u),v),\qquad \forall v\in H_0^1(\Omega),t\in J. \end{aligned}\tag{2.1}$$

Let μ =span$(N_1,\cdots,N_m)\subset H_0^1(\Omega)$ be the finite element space associated with a quasi-regular polygonalization of Ω such that the elements have diameters bounded by h, let the index of μ be the integer k. Let $\lambda>\frac{1}{4}(a^*/q_*)$ be a constant, $c_{nl}(U)=c(t_n,t_{l+\frac{1}{2}},x,U^{l+\frac{1}{2}})$, and $\phi^n(U)=\phi(x,t_n,U^n)$ for $\phi=q,a,b,p,r,f$. A Galerkin scheme is obtained by finding $U^{n+1},W^{n+1}\in\mu$ such that

$$\begin{aligned} &(q^n\partial_t W^n,v)+\lambda(\Delta t)^2(q^n\nabla\partial_{tt}W^n,\nabla v)\\ &+(a^n(U)\nabla W^n+b^n(U)\nabla U^n+\Delta t\sum_{l=0}^{n-1}c_{nl}(U)\nabla U^{l+\frac{1}{2}},\nabla v)\\ &-(p^n(U)\nabla W^n+r^n(U)\nabla U^n,v)\\ &=(f^n(U),v)+(\tilde{q}^n E\partial_t W^n,v),\qquad \forall v\in\mu,\\ &d_tU^n=W^{n+\frac{1}{2}},\qquad n=3,4,5,\cdots. \end{aligned}\tag{2.2}$$