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ROBUSTNESS OF AN UPWIND FINITE DIFFERENCE SCHEME FOR SEMILINEAR CONVECTION-DIFFUSION PROBLEMS WITH BOUNDARY TURNING POINTS *

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Abstract

We consider a singularly perturbed semilinear convection-diffusion problem with a boundary layer of attractive turning-point type. It is shown that its solution can be decomposed into a regular solution component and a layer component. This decomposition is used to analyse the convergence of an upwinded finite difference scheme on Shishkin meshes.

Key words: Convection-diffusion, Singular perturbation, Solution decomposition, Shishkin mesh.

1. Introduction

We consider the singularly perturbed semilinear convection-diffusion problem

$$\mathcal{T}u(x) := -\varepsilon u''(x) - x^p a(x)u'(x) + x^p b(x,u(x)) = 0 \quad \text{for} \quad x \in (0,1), \tag{1a}$$

$$u(0) = \gamma_0,\ u(1) = \gamma_1, \tag{1b}$$

where $0 < \varepsilon \ll 1$ is a small constant, $p > 0$, $a(x) > \alpha > 0$, $b_u \geq 0$ for $x \in [0,1]$, $a \in C^1[0,1]$ and $b \in C^1([0,1] \times I\!R)$. Its solution u typically has a boundary layer of width $\mathcal{O}\big(\varepsilon^{1/(p+1)} \ln \varepsilon\big)$ at $x = 0$. Numerical schemes for the case when $p = 0$ have been extensively studied in the literature; see [6] for a survey.

The class of problems considered includes

$$-\varepsilon u'' - xu' + xu = 0, \quad \text{for} \quad x \in (0,1), \quad u(0) = \gamma_0,\ u(1) = \gamma_1,$$

which models heat flow and mass transport near oceanic rises [1]. Multiple boundary turning points ($p > 1$) are also covered by (1) and they too arise in applications [7].

We are aware of four publications that analyse numerical methods for (1) with $p = 1$. Liseikin [2] constructs a special transformation and solves the transformed problem on a uniform mesh. The method obtained is proven to be first-order uniformly convergent in the discrete maximum norm. Vulanović [8] studies an upwind-difference scheme on a layer-adapted Bakhvalov-type mesh and proves convergence in a discrete ℓ_1 norm. This result is generalized in [9] for quasilinear problems. In [3] the authors establish almost first-order convergence in the discrete ℓ_∞ norm for an upwind difference scheme on a Shishkin mesh. There are also a number of papers that consider problems of the type

$$-\varepsilon u''(x) - x^p a(x)u'(x) + c(x,u(x)) = 0 \quad \text{in} \quad (0,1)$$

with Dirichlet boundary conditions and $c_u(0,u(0)) \geq \gamma > 0$. In this case, however, the behaviour is dominated by the relation between the diffusion term and the reaction term. The layer structure is like that of reaction-diffusion problems and is different from the layer occurring in (1). We are not aware of any publication that considers numerical methods for (1) with general $p > 0$.

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The main purpose of the present paper is to derive a decomposition of the solution of (1) into a regular solution component and a boundary layer component, with sharp estimates for their derivatives up to the third order (Section 2). In Section 3 we shall show how this decomposition can be used to analyse the convergence of an upwinded difference scheme for the approximate solution of (1). We prove that the scheme on a Shishkin mesh is almost first-order convergent in the discrete maximum norm, no matter how small the perturbation parameter ε may be. This error analysis is based on a hybrid stability inequality derived in [3] which implies that the error in the ℓ_∞ norm is bounded by a specially weighted ℓ_1 norm of the truncation error.

Notation. By C we denote throughout the paper a generic positive constant that is independent of ε and of N, the number of mesh nodes used.

2. Solution Decomposition

Theorem 1. *Let $a \in C^1[0,1]$ and $b \in C^1([0,1]\times I\!R)$. Then (1) has a unique solution $u \in C^3[0,1]$ and this solution can be decomposed as $u = v + w$, where the regular solution component v satisfies*

$$\mathcal{T}v = 0, \quad \left|v'(x)\right| + \left|v''(x)\right| \le C \quad \text{and} \quad \varepsilon\left|v'''(x)\right| \le Cx^p \quad \text{for} \quad x \in (0,1),$$

while the boundary layer component w satisfies

$$\tilde{\mathcal{T}}w := -\varepsilon w'' - x^p a w' + x^p \tilde{b}(x,w) = 0, \quad \tilde{b}(x,w) = b(x, v+w) - b(x,v)$$

and

$$\left|w^{(i)}(x)\right| \le C\mu^{-i}\exp\left(-\frac{\alpha x^{p+1}}{\varepsilon(p+1)}\right) \quad \text{for} \quad i = 0,1,2,3, \quad x \in (0,1)$$

with $\mu = \varepsilon^{1/(p+1)}$.

Proof. The decomposition is constructed as follows. We define v and w to be the solution of the boundary-value problems

$$\mathcal{T}v = 0 \ \text{ for } \ x \in (0,1), \quad a(0)v'(0) = b(0, v(0)), \ v(1) = \gamma_1 \tag{2a}$$

and

$$\tilde{\mathcal{T}}w = 0 \ \text{ for } \ x \in (0,1), \quad w(0) = \gamma_0 - v(0), \ w(1) = 0. \tag{2b}$$

The bounds for v and w and their derivatives will be given in Sections 2.2 and 2.3.

2.1. Preliminaries

Let

$$A(x) := \frac{1}{\varepsilon}\int_0^x s^p a(s) ds$$

and choose α^* to satisfy $a(x) \ge \alpha^* > 0$. For our analysis we need bounds for a number of integral expressions involving A. First of all we have

$$-A(x) \le -\frac{\alpha^*}{\varepsilon}\frac{x^{p+1}}{p+1} \quad \text{and} \quad A(s) - A(x) \le \frac{\alpha^*}{\varepsilon}\frac{s^{p+1} - x^{p+1}}{p+1} \quad \text{for} \quad 0 \le s \le x \le 1. \tag{3}$$

From this, for arbitrary $q \ge 0$ we get

$$\frac{\alpha^*}{\varepsilon}\int_0^x s^{(p+q)}\exp(A(s) - A(x))ds \le \frac{\alpha^*}{\varepsilon}\int_0^x s^p \exp\left(\frac{\alpha^*}{\varepsilon}\frac{s^{p+1} - x^{p+1}}{p+1}\right)ds \le 1. \tag{4}$$

We shall also use

$$\begin{aligned}\int_0^1 \exp(-A(s))ds &\ge \int_0^1 \exp\left(-\frac{\|a\|_\infty s^{p+1}}{(p+1)\varepsilon}\right)ds = \mu\int_0^{1/\mu}\exp\left(-\frac{\|a\|_\infty t^{p+1}}{(p+1)}\right)dt \\ &\ge \mu\int_0^1 \exp\left(-\frac{\|a\|_\infty t^{p+1}}{(p+1)}\right)dt = C\mu.\end{aligned} \tag{5}$$