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DISCRETE MINUS ONE NORM LEAST-SQUARES FOR THE STRESS FORMULATION OF LINEAR ELASTICITY WITH NUMERICAL RESULTS *1)

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Abstract

This paper studies the discrete minus one norm least-squares methods for the stress formulation of pure displacement linear elasticity in two dimensions. The proposed least-squares functional is defined as the sum of the L^2- and H^{-1}-norms of the residual equations weighted appropriately. The minus one norm in the functional is replaced by the discrete minus one norm and then the discrete minus one norm least-squares methods are analyzed with various numerical results focusing on the finite element accuracy and multigrid convergence performances.

Key words: H^{-1} least-squares, Linear elasticity, Multigrid method.

1. Introduction

In recent years there has been an increased interest in the use of least-squares methods for numerical approximation of the incompressible Stokes and Navier-Stokes equations [3, 4, 5, 6, 7, 11, 12] and for linear elasticity equations [9, 10, 11, 12, 13, 18, 21]. Such least-squares approaches are known as including accurate approximations to meaningful physical quantities, formulation of a well-posed minimization principle and freedom in the choice of finite element spaces which are not subject to the LBB condition.

In this paper, we attempt to apply H^{-1} least-squares method to planar linear elasticity equations with pure displacement boundary conditions:

$$\left\{ \begin{array}{rcll} -\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} & = & \mathbf{f} & \text{in} \quad \Omega, \\ \mathbf{u} & = & \mathbf{0} & \text{on} \quad \partial\Omega, \end{array} \right. \tag{1.1}$$

where Ω is a bounded open connected domain in $\Re^2$ with Lipschitz boundary $\partial\Omega$; $\mathbf{u}$ denotes the displacement; $\mathbf{f}$ is a given body force; and μ, $\lambda > 0$ are the Lamé constants. We assume that the elastic material is isotropic, homogeneous, and strongly elliptic. Denote by the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)} \in (0, \frac{1}{2})$.

It is well known that standard Galerkin finite element formulations for elasticity problem using piecewise linear elements are accurate for moderate values of a Lamé constant λ, but, as the elastic material becomes nearly incompressible, i.e. as $\lambda \to \infty$ (or $\nu \to \frac{1}{2}$), their approximation properties degrade severely [1, 20]. To overcome this, so-called *locking phenomenon*, Cai, Manteuffel, McCormick and their coworkers have developed recently first-order system

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L^2-norm and H^{-1}-norm least squares methods with the flux and vorticity formulation for the generalized Stokes equations that apply to the pure displacement problem of linear elasticity in [11, 13], and for the pure traction problem in [12]. In our formulation, defining new two variables as the strain tensor $\underset{\sim}{\boldsymbol{\sigma}} = \sqrt{2\mu}\underset{\sim}{\boldsymbol{\epsilon}}(\mathbf{u})$ scaled by $\sqrt{2\mu}$ and pressure $p = -\nabla \cdot \mathbf{u}$, the second-order pure displacement problem is reduced to first-order system of linear equations, so-called strain-displacement-pressure formulation. In the analysis of structural mechanics, the knowledge of the stress or strain is often of greater interest than the knowledge of the displacement. Even though the approximation of the stress or strain can be recovered from the displacement by postprocessing in the standard finite element formulation, in a numerical point of view their computations require the derivatives of the displacement which imply a loss of precision. But, in the strain-displacement-pressure formulation we used, the accurate strain can be obtained directly and the stress can be directly recovered as the linear combination $-\lambda p\, \mathrm{I}_2 + \sqrt{2\mu}\, \underset{\sim}{\boldsymbol{\sigma}}$ of strain tensor $\underset{\sim}{\boldsymbol{\sigma}}$ and pressure p where I_2 is 2×2 identity matrix. The similar formulation for the elasticity problem can be found in [21] and the stress formulation for the incompressible Stokes equations were applied to the mixed methods and stabilized Galerkin methods in [2, 15, 17].

Our least-squares functional is similar to that in [3] with $q = -1$ but it is appropriately weighted by a Lamé constant μ. We will directly establish ellipticity and continuity of the functional in a product norm involving Lamé constants μ and λ and the L^2- and H^1-norms. To make the computation of H^{-1}-norm to be feasible, we replace the H^{-1}-norm in the functional by the discrete H^{-1}-norm following the idea proposed by Bramble, Lazarov and Pasciak including discrete H^{-1}-norm least-squares approaches for scalar second-order elliptic equations in [6] and for the Stokes equations in [7]. Such discrete H^{-1} functional is shown to be uniformly equivalent to the Sobolev norms weighted by the Lamé constants. From this property we show that standard finite element discretization error estimates are optimal with respect to the order of approximation as well as the required regularity of the solution, and that they are uniform in the Lamé constants.

The paper is organized as follows. In section 2, we formulate an equivalent first-order systems with the strain-displacement-pressure formulation to pure elasticity problem and set some preliminary results. We introduce H^{-1}-norm least-squares functional weighted appropriately by Lamé constant μ for the strain formulation and then we establish its ellipticity and continuity in section 3. In section 4, we consider discrete H^{-1}-norm least-squares functional and discuss an error estimate according to [6] and [7]. Finally section 5 investigates a preconditioner for the resulting algebraic linear system and present the numerical results implemented by preconditioned Richardson iteration method and multigrid V-cycle algorithm using continuous piecewise linear finite element spaces.

2. First-Order System Formulations

In this section we formulate a first-order system for H^{-1} least-squares methods with the strain formulation that is equivalent to the system of equations of linear elasticity with pure displacement boundary conditions.

For convenience, we let the boldface denote the vector valued function and the under tilde boldface the matrix-valued function, i.e., the tensor. We use C with or without subscripts to denote a generic positive constant, possibly different at different occurrences, that is independent of the Lamé constants and other parameters introduced in this paper, but may depend on the domain Ω. The colon notation : denotes the inner product on $\Re^{2\times 2}$ and for any tensors $\underset{\sim}{\boldsymbol{\tau}} = (\tau_{ij})$ and $\underset{\sim}{\boldsymbol{\delta}} = (\delta_{ij})$ in $L^2(\Omega)^{2\times 2}$, the $L^2(\Omega)^{2\times 2}$ inner product is defined by

$$(\underset{\sim}{\boldsymbol{\tau}}, \underset{\sim}{\boldsymbol{\delta}}) = \int_\Omega \underset{\sim}{\boldsymbol{\tau}} : \underset{\sim}{\boldsymbol{\delta}}\, dx.$$