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SYMMETRIC POINT STRUCTURE OF SUPERCONVERGENCE FOR CUBIC TRIANGULAR ELEMENTS -A CONSULTATION WITH ZHU *1)

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Abstract

Superconvergence structures for rectangular and triangular finite elements are summarized. Two debatable issues in Zhu's paper [18] are discussed. A superclose polynomial to cubic triangular finite element is constructed by area coordinate.

Key words: Cubic triangular element, Superconvergence, Symmetric points.

1. Summary on Superconvergence Structures

Suppose that domain Ω is a square with the boundary Γ and triangulation J^h in Ω is uniform. We shall discuss n-degree triangular family $P_n = \sum_{i+j\leq n} b_{ij}x^iy^j$ and n-degree rectangular family Q_n. Denote by $S_0^h = \{v \in H^1(\Omega), v|_\tau \in P_n(\text{or } Q_n),\ \tau \in J^h,\ v = 0 \text{ on } \Gamma\}$ the n-degree finite element subspace. The solution $u \in H_0^1(\Omega)$ of second order elliptic problem and its finite element approximation (Ritz-projection) $u_h \in S_0^h$ satisfy the orthogonal relation

$$A(u - u_h, v) = 0, \quad v \in S_0^h, \tag{1}$$

where the bilinear form $A(u,v) = \int_\Omega (a_{ij}D_iuD_jv + a_{00}uv)dx$ is assumed to be bounded and H_0^1-coercive. Denote by $W^{k,p}(\Omega)$ Sobolev space with norm $||u||_{k,p,\Omega}$. If $p = 2$, simply use $H^k(\Omega)$ and $||u||_{k,\Omega}$. It is well known that there are the error estimates

$$||u - u_h||_{l,\infty,\Omega} = O(h^{n+1-l}\ln h),\ l = 0, 1. \tag{2}$$

But, u_h or Du_h at some specific points possibly possess the higher rate of convergence (called superconvergence by Douglas).

In the conference on superconvergence in finite element method on March 15-30, 2000, at Berkeley, two chairmen Babuska and Wahlbin claimed that there are three present schools of superconvergence, i.e Ithaca (Locally symmetry theory [12,13,14]), Texas (Method based on the computer [1,2]) and China (Element othogonality analysis, see [6,7,11]). In another conference on three-dimensional finite elements on August 2000 at Jyvaskyla, Brandts and Krizek [3] also summarized three different methods of three schools.

From numerous researches on superconvergence up to now, we know that there are two basic structures of superconvergcne, i.e. Gauss-Lobatto points and symmetric points. Firstly, for regular rectangular element $u_h \in Q_\lambda(n) = \sum_{(i,j)\in I_{n,\lambda}} b_{ij}x^iy^j$, where $I_{n,\lambda} = \{(i,j)|0 \leq i,j \leq n, i+j \leq n+\lambda\}, 1 \leq \lambda \leq n$, we early known [5,6,10,19] that u_h and its gradient Du_h have superconvergence at $n+1$-order Lobatto points and n- order Gauss points, respectively. Besides, if n is odd, the average gradient $\bar{D}u_h$ has superconvergence at vertexes and n-order Gauss points on each side of the element. Secondly, if the number of parameter is decreased, it reduces to the rectangular defective (or serendipity) family $Q'(n) = P_n \oplus span\{x^ny, xy^n\}$ and

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n-degree triangular family $P_n = \sum_{i+j\le n} b_{ij}x^iy^j$. At this time, u_h (for even n) and the average gradient $\bar{D}u_h$ (for odd n) have superconvergence at symmetric points T_h, where T_h consists of four vertexes, four side midpoints and center for rectangular element (see [1,2,9,15]), and three vertexes and three side midpoints for triangular element (see [1,2, 6,7,8,12,13,14,16]).

Here, an interesting topic for us is that whether there exist other superconvergence points for triangular elements, besides symmetric points. We should point out that Wahlbin [12,13,14] first time proved superconvergence at locally symmetric points in quite extensive framework. Of course, their paper has not given the answer to the question mentioned above. However, Babuska [1,2] have calculated the derivative D_xu_h in a triangle for $1 \le n \le 7$ based on the computer and have pointed out that the midpoint of a side parallel to x-axis is only superconvergence point for D_xu_h (but the averaging have not been considered) for $n = 1, 3, 5, 7$, and have found no other points. And no superconvergence point of D_xu_h for $n = 4, 6$, but, $n = 2$ is an exceptional case, 2-Gauss points on this side are superconvergent. Recently, Babuska and Strouboulis have depicted a fig. $4.7^*.8$ for D_xu_h of the cubic triangular element in their new book [2] and especially emphasized that "Note that the mid-points of the sides which are parallel to the x-axis are the only superconvergence points in the case of the Poisson equation". We also proved [8] that $\bar{D}_xu_h$ for cubic triangular element u_h has no superconvergence points, besides symmetric points, and there are no superconvergence points for u_h itself at all. We exhibit the numerical examples in a square $\Omega = \{0, x, y < 1\}$ as follows.

Consider an elliptic problem $-\triangle u = f$ in Ω, $u = 0$ on $\Gamma_0 = \{x = 0, 0 < y < 1\} \cup \{y = 0, 0 < x < 1\}$ and $D_nu = 0$ on $\Gamma_1 = \{x = 1, 0 < y < 1\} \cup \{y = 1, 0 < x < 1\}$. The exact solution $u = (13x - 8x^2 + x^3)(2y - y^2)$. Ω is subdivided into regular triangular uniform meshes J^h, $h = 1/N, N = 4, 8$. We have calculated the cubic finite element u_N and its error $e_N = u - u_N$ in the following table 1.

The table 1. The error e_4, e_8 at nodes and the ratio $e_4 : e_8$

	$x = 1/4$	1/2	3/4	1
$y = 1/4$	2.389E-4	1.913E-4	1.669E-4	1.436E-4
	1.207E-5	1.117E-5	9.489E-6	8.393E-6
	18.86	17.13	17.57	17.11
$y = 1/2$	2.399E-4	1.935E-4	1.699E-4	1.593E-4
	1.393E-5	1.243E-4	1.074E-5	9,521E-6
	17.22	15.57	15.82	16.73
$y = 3/4$	2.680E-4	2.142E-4	1.876E-4	1.783E-4
	1.503E-5	1.355E-5	1.187E-5	1.063E-5
	17.83	15.81	15.80	16.77
$y = 1$	2.510E-4	2.200E-4	1.910E-4	3.630E-4
	1.501E-5	1.421E-5	1.252E-5	2.274E-6
	15,78	15.48	15.26	16.00

We see that when triangulation is refined twice, the error ratio $e_4 : e_8 = 15.3 \sim 18.9$, thus the cubic triangular element has only the accuracy $O(h^4)$ at nodes, no superconvergence. A detailed data analysis shows that its accuracy at nodes is the worst. The facts mentioned above show that the cubic triangular elements do not possess Gauss-Lobatto point structure of superconvergence, which is of a great difference from the regular rectangular elements.

2. Discussion on Zhu's paper [18]

Early Chen [4] proved by the element analysis that the average gradient $\bar{D}u_h$ for triangular linear element has superconvergence at six symmetric points in each triangular element. Later, Zhu [16] proved by this method that the quadratic triangular element u_h itself has superconvergence at six symmetric points. Although the natural superconvergcne points within an element