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SOLUTION OF BURGERS' EQUATION USING THE MARKER METHOD

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Abstract. A new method for the solution of Burgers' equation is described. The marker method relies on the definition of a convective field associated with the underlying partial differential equation; the information about the approximate solution is associated with the response of an ensemble of markers to this convective field. Some key aspects of the method, such as the selection of the shape function and the initial loading, are discussed in some details. The marker method is applicable to a general class of nonlinear dispersive partial differential equations.

Key Words. Particle method, Burger equation, marker method, shape function

1. Introduction

Marker methods have been used for a long time in various disciplines (e.g plasma physics, astrophysics, etc.) to give numerical solution of purely convective problems [1, 2]. In these methods an ensemble of markers (or 'superparticles') is used to approximate the solution; the region of interest covered by the markers defines the phase space associated with the solution. Each marker is represented through its weight and position in phase space. The markers are advanced in time according to the characteristics ('equations of motion') of the underlying partial differential equation (PDE) associated with the problem. Marker methods are particularly useful for collisionless problems [1, 2, 3, 6]. However, in many applications of interest (e.g turbulent plasmas), diffusive processes can be important. Marker methods usually include diffusive effects in a perturbative fashion [4, 5]: in the first step, the markers are evolved in phase space according to the collisionless (*i.e.* purely convective) dynamics; in the second step, diffusive effects are included by a randomization of the markers' weights and/or positions according to a prescribed probability distribution. Although this method agrees with physical intuition, it is, from the numerical point of view, quite noisy and possibly inaccurate. The marker method presented in this paper allows for the *simultaneous* treatment of convective and diffusive effects.

The main idea behind the marker method for the solution of a given PDE is to rewrite it as a conservation equation with a generalized convective velocity. In general (even in linear cases), the generalized convective velocity depends on the solution of the PDE itself. Each marker, which carries the information of the solution of the PDE through its weight and its position, is advanced in time using a

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Lagrangian scheme. The generalized convective velocity mentioned earlier is computed through the information contained in the ensemble of markers and through the so-called shape function.

As it will become apparent in the next sections, the marker method can actually be applied to solve a more general class of PDEs that are encountered commonly in physical and engineering sciences.

The marker method, unlike the finite difference and the finite element methods, does not rely on the concept of a grid (of course one can, if needed, reconstruct the solution on a fixed grid through the collective information associated with the markers). Increased resolution can be achieved in a natural way by locally increasing the number of markers and/or modifying the initial loading of the markers. Unlike the finite difference method, the marker method can be trivially extended to multi-dimensional problems.

This paper is organized as follows; in section 2, the marker method is described in the context of the solution of a one-dimensional linear diffusion equation. The shape function, which is involved in the evaluation of the approximate solution, is analyzed in some detail and a numerical example is presented. The marker method is applied to the nonlinear Burgers' equation [8] in section 3. Concluding remarks are given in section 4.

2. Marker Method

The purpose of this paper is to present a new numerical method for the solution of the nonlinear Burgers' equation [8]

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} = \mu\frac{\partial^2 f}{\partial x^2} \,, \tag{1}$$

with initial conditions $f(x,0) = f_0(x)$ and $\mu > 0$ is a constant. As mentioned in the Introduction, particle methods are usually applied to purely convective problems [*i.e.* by neglecting the right-hand side in Eq.(1)]. Therefore, the new aspect of the marker method is best described in the context of a simple example: the linear diffusion equation which is a limiting case of Eq.(1). An analysis of the smoothing approximation obtained through the shape function, which represents a crucial aspect of the method, is also discussed in this section. A specific numerical application of the marker method to the case of a one-dimensional linear diffusion equation is given.

2.1. Basic Idea. For illustrative purposes, we describe the marker method for one-dimensional problems (as mentioned in the Introduction, the generalization to multi-dimensional problems is straightforward). We consider an ensemble of N markers. Each marker k is defined through its position x_k and its weight W_k. The solution of a given one-dimensional PDE is found by allowing the set $\{(x_k, W_k); k = 1, \cdots, N\}$ to evolve in time according to a generalized nonlinear convective velocity. The generalized convective velocity usually depends on the solution itself and a form of convolution of the approximate solution with a shape function is required.
Consider the one-dimensional diffusion equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \,, \tag{2}$$