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A NOTE ON VECTOR CASCADE ALGORITHM*[1)]

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Abstract

The focus of this paper is on the relationship between accuracy of multivariate refinable vector and vector cascade algorithm. We show that, if the vector cascade algorithm (1.5) with isotropic dilation converges to a vector-valued function with regularity, then the initial function must satisfy the Strang-Fix conditions.

Key words: Cascade algorithm, Accuracy, Symbol, Refinable vector.

1. Introduction and Main Result

Fox a fixed integer d no less than 1, let A be a $d \times d$ matrix with integer entries and all eigenvalues of modulus > 1.

In wavelet theory, we are often concerned with functional equation of the form

$$\Phi(x) = \sum_{k \in Z^d} c_k \Phi(Ax - k), \tag{1.1}$$

where Φ is the unknown vector of functions defined on the d-dimensional Euclidean space R^d and $c = \{c_k\}_{k \in Z^d}$ is a finitely supported $r \times r$ matrix sequence on Z^d. We call the equation (1.1) refinement equation. Any vector-valued function satisfying a refinement equation is called a refinable vector. In Fourier domain, (1.1) is equivalent to

$$\hat{\Phi}(\xi) = m\left(\left(A^T\right)^{-1}\xi\right)\hat{\Phi}\left(\left(A^T\right)^{-1}\xi\right), \tag{1.2}$$

where $m(\xi) = |\det A|^{-1} \sum_{k \in Z^d} c_k e^{-ik\xi}$.

The matrix A is called a dilation matrix, the sequence c is called a refinement mask and $m(\xi)$ is called a symbol.

The equation (1.1) is the starting point for construction of wavelets (in scalar case) and multiwavelets (in multiple case) in one dimensional or higher-dimensional case (see [5], [2], [8], [14] et al). The usual choice of A is 2 or $2I_{d\times d}$ (the $d \times d$ identity matrix). Recently, the isotropic matrix dilations have already been considered in [2] and [14] to construct non-separable bidimensional wavelet bases. For example, in [2] A is chosen to be $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and

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$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. Here we say A is isotropic if A is similar to a diagonal matrix $\text{diag}(\sigma_1, \cdots, \sigma_d)$ with $|\sigma_1| = \cdots = |\sigma_d| = |\det A|^{\frac{1}{d}}$. In this case, there exists an invertible matrix Λ such that

$$\Lambda A \Lambda^{-1} = \text{diag}\,(\sigma_1, \cdots, \sigma_d)\,. \tag{1.3}$$

Obviously, the dilation matrices studied in [2] are isotropic.

The existence of the solution of the equation (1.1) has been studied by many mathematicians ([6], [9], [1]). It is well-known that there is no closed-form analytic formula even for scalar univariate orthogonal refinable functions in the case with 2 dilation. The cascade algorithms are often used to study the solutions of the refinement equation (1.1). Associated with refinement equation (1.1), we define refinement operator

$$Q_c f(\cdot) := \sum_{k \in Z^d} c_k f(A \cdot - k), \tag{1.4}$$

for each component of the vector-valued function f belongs to $L_q(R^d)$, the familiar space $-$ the set of all q-absolutely integrable functions in R^d.

Let Φ_0 be an initial vector-valued function with compact support in $L_q(R^d)$. For $n = 1, 2, \cdots$, define

$$\Phi_n := Q_c \Phi_{n-1}. \tag{1.5}$$

Clearly $\Phi_n = Q_c^n \Phi_0$, $n = 0, 1, \cdots$ (here we let $Q_c^0 \Phi_0 = \Phi_0$). The algorithm (1.5) is called cascade algorithm with mask c and dilation A.

Convergence of cascade algorithms has been studied by many mathematicians ([3], [11], [15]). In this paper, we are interested in vector cascade algorithms with isotropic dilation. We shall prove that, in order to get a refinable vector of functions with higher accuracy and smoothness, the initial vector-valued function Φ_0 must satisfy the Strang-Fix conditions.

We say that the vector-valued function Φ has accuracy p if all multivariable polynomials with total degree less than p can be reproduced from linear combinations of multi-integer translates of the functions $\phi_1, \cdots, \phi_r$, or equivalently, $\Pi_{p-1} \subset S(\Phi)$, where Π_{p-1} denotes the linear space of all polynomials in d variable of total degree at most $p-1$ and $S(\Phi) = \{f : f = \sum_{l \in Z^d} \sum_{k=1}^{r} a_{kl} \phi_k(\cdot - l)\}$ with any sequence $\{a_{kl}\}$. We say $S(\Phi)$ is principal shift-invariant (PSI) if $r = 1$ and finite shift-invariant (FSI) if $r > 1$.

In wavelet theory, accuracy is a desirable property of scaling function. The accuracy p of orthonormal refinable function leads to the p vanishing moments of wavelets which is very important for image compression.

A single compactly supported function $f \in L_1(R^d)$ is said to satisfy the Strang-Fix conditions of order p if

$$\hat{f}(0) = 1 \quad \text{and} \quad D^\mu \hat{f}(2\pi k) = 0, \quad \text{for} \quad 0 \le \mu, |\mu| < p, k \in Z^d / \{0\}. \tag{1.6}$$

A vector-valued function $\Phi = (\phi_1, \cdots, (\phi_r)^T$ is said to satisfy the Strang-Fix conditions of order p if there exists a scalar function $\psi \in S(\Phi)$ satisfying the usual Strang-Fix conditions (1.6).

It is stated in [12] that any vector-valued function (refinable or not refinable) has accuracy p if and only if the vector of functions satisfies the Strang-Fix conditions of order p under the assumption of linear independence. For refinable functions, it is crucial to explore the conditions of the mask c and the symbol $m(\xi)$ when refinable vector Φ has accuracy p.

In the case of a single, one-dimensional refinable function, $A = 2$, the requirement for Φ to have accuracy p is the following set of sum rules:

$$\sum_{k=0}^{N} c_k = 2 \qquad \text{and} \qquad \sum_{k=0}^{N} (-1)^k k^j c_k = 0, \quad \text{for} \;\; j = 0, \cdots, p-1.$$