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STRUCTURE-PRESERVING ALGORITHMS FOR DYNAMICAL SYSTEMS*

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Abstract

We study structure-preserving algorithms to phase space volume for linear dynamical systems $\dot{y} = Ly$ for which arbitrarily high order explicit symmetric structure-preserving schemes,i.e. the numerical solutions generated by the schemes satisfy $\det(\frac{\partial y_1}{\partial y_0}) = e^{htrL}$, where trL is the trace of matrix L, can be constructed. For nonlinear dynamical systems $\dot{y} = f(y)$ Feng-Shang first-order volume-preserving scheme can be also constructed starting from modified θ - methods and is shown that the scheme is structure-preserving to phase space volume.

Key words: structure-preserving algorithm, phase space volume, source-free dynamical system.

1. Introduction

Consider the ODE_S

$$\frac{dy}{dt} = f(y),\ y \in \Re^n \tag{1.1}$$

with solutions $y(t)$ and Jacobian $B(t) = \frac{\partial y(t)}{\partial y(o)}$ which satisfies the initial problem

$$\begin{cases} \frac{d}{dt}B(t) = FB \\ B(o) = I, \end{cases}$$

where $F(y) = df(y)$ is the derivative of the vector field f. We have

$$\frac{d}{dt}\det B(t) = \det B(t) tr(B^{-1}\frac{d}{dt}B) = \det B(t)\ trF$$

so that phase space volume contracts, conserves or expands when $trF < 0,\ trF = 0$ or $F > 0$ for all y respectively. trF is divergence or trace of the vector field f. Up to now in the field of numerical integration, much work [2]-[7],[9] has been done in maintaining the preservation of phase space volume for source-free dynamical systemes:

$$\frac{dy}{dt} = f(y),\ y \in \Re^n \tag{1.2}$$

which satisfy

$$(div\ f)(y) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial y_i} = 0. \tag{1.3}$$

Definition 1.1. In numerically solving a source-free system (1.2)-(1.3), an one-step scheme is called volume-preserving scheme if, as applied to the source-free systems, the numerical solutions generated by the one-step scheme satisfy the volume-preserving condition

$$\det\left(\frac{\partial y_1}{\partial y_0}\right) = 1. \tag{1.4}$$

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Already for linear source-free systems of dimension $n \geq 3$, a key Lemma 1 in [2] shows that no standard method can be volume-preserving. This negative result motivates the search for new methods which can maintain the preserving of phase space volume. Up to the present the following approaches are found: First the splitting idea yields an approach, for example, proposed by K.feng & Z.J.Shang [2] which comes from decomposing a source-free voctor field as a finite sum of 2-dimensional Hamcltionian fields for which symplectic Euler formula is used. Second approach comes from using generating function [5]-[7].

For standard method, for example, Runge-Kutta method and so on only some special source-free systems have been discussed [4],[9].

This paper is organized as follows. In Section 2, we aim at perfecting the work of [4]. It is shown that for some special source-free systems symmetric and symplectic Runge-Kutta methods as well as symmetric partitioned RK methods with $\overline{b_i} = b_i, i = 1, \cdots, s$ are volume-preserving. In Section 3, for a general linear dynamical system $\dot{y} = Ly$ first doing exponential transformation, and applying a modified θ-method to the new system generated by the transformation can yield some first-order explicit structure-preserving schemes which satisfy

$\det(\frac{\partial y_1}{\partial y_0}) = e^{htrL}$,and then we compose the first-order schemes into arbitrarily high order explicit symmetric structure-preserving ones. In Section 4, for non-linear dynamical systems $\dot{y} = f(y)$ Feng-Shang first-order volume-preserving scheme can be also constructed starting modified θ - methods and is shown that the scheme is structure-preserving to phase space volume.

2. Volume-preserving scheme for linear system with canonical form

As an example, already discussing in all details [2] linear source-free system in $\Re^3$

$$\begin{cases} \dfrac{dy}{dt} = Ly, & (2.1) \\ trL = 0 & (2.2) \end{cases}$$

solved by trapezied formula will give us valuable enlightenment. First we can get algorithmic approximation G^h to $e_L^h = \exp(hL)$ with

$$G^h = (I - \frac{h}{2}L)^{-1}(I + \frac{h}{2}L),$$

in general, it is scarcely possible that $\det(G^h) = 1$. Now we pay attention to the following fact of the matter:

For matrix $L = (l_{ij})_{i,j=1}^3$ which satisfies $trL = 0$, there exists a reversible matrix P such that the value of $P^{-1}LP$ only takes one of

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where $\sum_{i=1}^{3} \lambda_i = 0$ and $\prod_{i=1}^{3} \lambda_i = \det(L)$. Obviously, only the first case $(0, -\lambda_1, \lambda_1)$ means that it is possible for the systems (2.1)-(2.2) to be stable, and leads to $\det(G^h) = 1$, i.e. trapezoid formula being volume-preserving. Orther case means that corresponding linear source-free systems (2.1)-(2.2) are not stable (that is, at least, there exists a component of solutions for systems (2.1)-(2.2) being unbounded as $t \to \infty$), and leads to trapezied formula being non-volume-preserving.

Now we extend such discussion to n-dimension linear source-free system.

In many application linear source-free systems (2.1)-(2.2) considered in $\Re^n$ are stable, so