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D-CONVERGENCE OF ONE-LEG METHODS FOR STIFF DELAY DIFFERENTIAL EQUATIONS*1)

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Abstract

This paper is concerned with the numerical solution of delay differential equations(DDEs). We focus on the error analysis of one-leg methods applied nonlinear stiff DDEs. It is proved that an A-stable one-leg method with a simple linear interpolation is D-convergent of order p, if it is consistent of order p in the classical sense.

Key words: Nonlinear delay differential equations, One-leg methods, D-convergence.

1. Introduction

In recent years, many papers discussed numerical methods for the solution of delay differential equation (DDE)

$$y'(t) = f(t, y(t), y(t-\tau)). \tag{1.1}$$

For linear stability of numerical methods , a significant number of results have already been found for both Runge-Kutta methods and linear multistep methods (cf.[4] [7] [8]).Recently, we further established the relationship between G-stability and nonlinear stability (cf.[3]). Error analysis of DDE solvers is another important issue. In fact, many papers investigated the local and global error behaviour of DDE solvers (cf.[1] [10]). However, error analysis of numerical methods for DDEs is mostly based on the fact that the function $f(t,y,z)$ satisfies Lipschitz conditions in both the last two variables. They are suitable for nonstiff DDEs because the Lipschitz constants are moderate. However, they can not be applied to stiff DDEs. For example, consider semi-discrete Hutchinson's equation (cf.[2]) with

$$f(t,y(t),y(t-\tau)) = \frac{a}{\Delta x^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ \vdots \\ y_N(t) \end{bmatrix} + \begin{bmatrix} y_1(t)(1-y_1(t-\tau)) \\ y_2(t)(1-y_2(t-\tau)) \\ \vdots \\ \vdots \\ y_N(t)(1-y_N(t-\tau)) \end{bmatrix}, \tag{1.2}$$

where $a > 0$ is the diffusion coefficient, $\Delta x = 1/(N+1)$ is a constant stepsize in space. In this case, the Lipschitz constant L of the function $f(t,y,z)$ with respect to y will contain negative powers of the meshwidth Δx in space. As a consequence, L will be very large for fine space grids, and the error estimates based on L are not realistic. On the other hand, the one-sided

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Lipschitz constant α is only moderate. Hence estimates based on α are often considerably more realistic than that based on L. Recently, the concept of D-convergence [11] for DDEs, which is a generalization of the concept of B-convergence (cf.[5] [6]) for ODEs, was introduced. In [3], we discussed D-convergence of A-stable one-leg methods with a complex interpolation procedure . In this paper, we further discuss D-convergence of A-stable one-leg methods with a more simple interpolation procedure.

2. Preliminaries

Let $\langle\cdot,\cdot\rangle$ be an inner product on C^N and $\|\cdot\|$ the corresponding norm. Consider the following nonlinear equation

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau)), \quad t \geq 0, \\ y(t) = \phi_1(t), \quad t \leq 0, \end{cases} \tag{2.1}$$

where τ is a positive delay term , ϕ_1 is a continuous function, and $f : [0, +\infty) \times C^N \times C^N \to C^N$, is a given mapping which satisfies the following conditions:

$$\mathrm{Re}\langle u_1 - u_2, f(t, u_1, v) - f(t, u_2, v)\rangle \leq \alpha\|u_1 - u_2\|^2, \quad t \geq 0, u_1, u_2, v \in C^N, \tag{2.2}$$

$$\|f(t, u, v_1) - f(t, u, v_2)\| \leq \beta\|v_1 - v_2\|, \quad t \geq 0, u, v_1, v_2 \in C^N, \tag{2.3}$$

where α and β are real constants. In order to make the error analysis feasible, we always assume that the problem (2.1) has a unique solution $y(t)$ which is sufficiently differentiable and satisfies

$$\|\frac{d^i y(t)}{dt^i}\| \leq M_i.$$

Remark 2.1. When $\beta = 0$, the above problem class has been used widely in stiff ODEs field (cf.[6]).

Before stating stability results, we introduce another system, defined by the same function $f(t, u, v)$, but with another initial condition:

$$\begin{cases} z'(t) = f(t, z(t), z(t-\tau)), \quad t \geq 0, \\ z(t) = \phi_2(t), \quad t \leq 0. \end{cases} \tag{2.4}$$

Proposition 2.2. *Suppose $\beta \leq -\alpha$.Then the following is true:*

$$\|y(t) - z(t)\| \leq \max_{x \leq 0} \|\phi_1(x) - \phi_2(x)\|, \quad t \geq 0. \tag{2.5}$$

The proof of this proposition can be found in [9]. Similarily, we can easily obtain the following result.

Proposition 2.3. *Suppose $\beta < -\alpha$. Then the following holds:*

$$\lim_{t \to +\infty} \|y(t) - z(t)\| = 0. \tag{2.6}$$

Now we consider the adaptation of one-leg methods to (2.1). We briefly recall the form of a one-leg method for the numerical solution of the ordinary differential equation

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \geq 0, \\ y(0) = y_0. \end{cases} \tag{2.7}$$

The one-leg k step method is the following

$$\rho(E)y_n = hf(\sigma(E)t_n, \sigma(E)y_n), \tag{2.8}$$

where $h > 0$ is the stepsize, E is the translation operator:$Ey_n = y_{n+1}$, each y_n is an approximation to the exact solution $y(t_n)$ with $t_n = nh$, and $\rho(x) = \sum_{j=0}^{k} \alpha_j x^j$ and $\sigma(x) = \sum_{j=0}^{k} \beta_j x^j$ are generating polynomials, which are assumed to have real coefficients, no common divisor .