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METHOD OF NONCONFORMING MIXED FINITE ELEMENT FOR SECOND ORDER ELLIPTIC PROBLEMS*1)

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Abstract

In this paper, the method of non–conforming mixed finite element for second order elliptic problems is discussed and a format of real optimal order for the lowest order error estimate.

Key words: Non–conforming mixed finite element, Error estimate, Second order elliptic problems.

1. Introduction

Recently Hiptmair (see[1]) and Farhloul & Fortin (see[2]) have constructed and analyzed some non–conforming finite element mixed methods for second order elliptic problems:

$$\begin{cases} -\mathrm{div}(a\nabla u) = f, & x \in \Omega, \\ u = 0, & x \in \partial, \end{cases} \tag{1.1}$$

where $\Omega \subset R^n$ $(n = 2, 3)$ is a bounded open field with Lipshitz continuous boundary $\partial\Omega$, f is a given function of the space $L^2(\Omega)$ and $a \in L^\infty(\Omega)$ is assumed to be uniformly positive and bounded:

$$0 < a_1 \le a(x) \le a_2, \ \ x \in \bar{\Omega}. \tag{1.2}$$

Introducing the auxiliary variable $p = a\nabla u$, the problems (1.1) may be written as the system:

$$\begin{cases} p - a\nabla u = 0, & x \in \Omega, \\ \mathrm{div} p = -f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Then the mixed variational formulation of (1.3) is:

Find $(p, u) \in H \times M$ such that

$$\begin{cases} a(p, q) + b(q, u) = 0, & \forall q \in H, \\ b(p, v) = -(f, v), & \forall v \in M. \end{cases} \tag{1.4}$$

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where $H = H(\mathrm{div};\Omega) = \{q \in L^2(\Omega)^n;\ \mathrm{div}q \in L^2(\Omega)$, $M = L^2(\Omega)$, $a(p,q) = (a^{-1}p, q)$, $b(q,v) = (\mathrm{div}q, v)$ and $(.,.)$ is the inner product in $L^2(\Omega)$ or $L^2(\Omega)^n$.

Let $\Im_h$ be a regular triangulation of $\bar{\Omega}$ (cf.[3]) and P_l be the space of polynomials of degrees less or equal to l (where $l \geq 0$ is an integer). The non–conforming discretization of the problem (1.4), constructed in [1] and [2], is to consider two finite–dimensional spaces H_h and M_h such that

1) There is an integer $k \geq 0$ such that $RT^k(\Im_h) \subset H_h$, where $RT^k(\Im_h)$ is the space of vector field arising from kth order Raviart –Thomas elements (see[4]).

2) The moments up to order $l(l \leq k)$ of the discrete flux are continuous across inter elements boundaries, i.e.

$$\int_e (q_{h|K_i}.n_i + q_{h|K_j} n_j) p_l \mathrm{d}s = 0, \ \ \forall p_l \in P_l.$$

for all internal faces $e = \partial K_i \bigcap \partial K_j (i \neq j)$ and all $q_h \in q_h \in H_h$ (where n_i denotes the unite outward normal on ∂K_i).

3) M_h has to satisfy the following condition: if $\forall q_h \in H_h$ and

$$\sum_{K \in \Im_h} \int_K \mathrm{div} q_h v_h \mathrm{d}x = 0, \ \ \forall v_h \in M_h,$$

then $\mathrm{div} q_{h|K} = 0, \ \ \forall K \in \Im_h$.

The non– conformity of this discretization is due to the fact that the discrete flux is not necessarily continuous across inter element boundaries. Hiptemair (see[1]) has proved the convergence and given error estimates for this non–conforming mixed finite elements for $k \geq l \geq 1$. His analysis is based so–called "Generalized Patch Test" (cf.[5]). Farhloul & Fortin have derived a non–conforming approximation of the lowest order in the two–dimensional case (see[2]). We have found that Farhloul & Fortin's format is not optimal as the approximation of the flux $p_{h|K} \in P_1(K)^2, \ \ \forall K \in \Im_h$, but its accuracy on L^2 norm is only $O(h)$. One knows that if $H_h \subset L^2(\Omega)^n, \ \ \forall q_h, p_h \in H_h, \ \ a(p_h, q_h)$ is continuous. Therefore, the error estimates of non–conforming mixed finite element are due to the estimates causing by bilinear forms $b(.,.)$. But in [1], the estimates of non–conforming element causing by $a(.,.)$ and $b(.,.)$ are all discussed. Thus, much work is in vain because $a(.,.)$ cannot cause the error estimates of non–conforming element.

In this paper, C denotes a positive constant independent of h, but may be inequality in different positions.

2. The Non–Conforming Element Analysis

Let $H_h \not\subset H, M_h$ be satisfied 1)–2) in the section 1. Then the discrete problem of (1.4) reads as follows:

Find $(p_h, u_h) \in H_h \times M_h$ such that

$$\begin{cases} a(p_h, q_h) + b_h(q_h, u_h) = 0, & \forall q_h \in H_h, \\ b_h(p_h, v_h) = -(f, v_h), & \forall v_h \in M_h, \end{cases} \tag{2.1}$$